The Evaluation of Connection Coefficients of Compactly Supported Wavelets

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Abstract

A *connection coefficient* is an integral of products of wavelet basis functions, their derivatives and their translates. It is a functional that can be used to solve PDE's by the wavelet-Galerkin method. This paper presents an exact method for evaluating connection coefficients. This is essential for the application of wavelets to the numerical solution of partial differential equations, since numerical approximations of the connection coefficients are in general unstable due to the oscillatory nature of the integrands.

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1 Introduction

Compactly supported wavelets have recently been applied to the numerical solution of nonlinear differential equations with encouraging results (cp. [5], [7]). These works employ the *wavelet-Galerkin* method, in which a wavelet series expansion for the unknown function is used with the traditional Galerkin method.

Any numerical method for solving a differential equation must be capable of approximating the derivatives and nonlinearities in the unknown function. This is straightforward for Fourier-based spectral methods because the basis functions are eigenfunctions of the operator of differentiation, and because products of the basis elements are also basis elements. For compactly supported wavelets, however, this is not the case. The approximation of the derivative and nonlinearities of a function in a wavelet basis reduces to the calculation of inner products of basis functions with products of differentiated basis functions.

For instance, in [5] we used the 6 coefficient Daubechies wavelet (D6) and the wavelet-Galerkin method to solve Burgers' equation. This required calculating integrals of the form

$$\int \varphi_l^{d_1}(x)\varphi_m^{d_2}(x)dx$$

for the difusion term, u_{xx} and

$$\int \varphi_k^{d_1}(x)\varphi_l^{d_2}(x)\varphi_m^{d_3}(x)dx$$

for the nonlinear term, uu_{xx}. These values (note that the superscripts d_i refer to differentiation) are what we refer to as 2-term and 3-term scaling function connection coefficients. The numerical approximation of these integrals are difficult since these integrals are highly oscillatory. In fact, numerical approximation is unnecessary: as we shall show, these integrals can be explicitly evaluated in finite terms.

The remainder of the paper is organized as follows: Section 2 introduces notation and terminology, Section 3 outlines our method, Section 4 presents the full method and Section 5 is devoted to questions concerning the existence and uniqueness of solutions. Appendix A demonstrates how to calculate moments of the scaling function. Appendix B presents the equations for the simpler 2-term case, and finally, in Appendix C we tabulate the connection coefficients for the Daubechies 6-coefficient wavelet basis D6, which suffice for the numerical solution of Burgers' equation.

2 Definition of Terms

2.1 Compactly Supported Wavelets

The class of compactly supported wavelet bases was introduced by I. Daubechies in 1988 [2]. They are an orthonormal basis for functions in $L^2(\mathbf{R})$. A "wavelet system" consists of the function $\varphi(x)$, referred to as the scaling function and the function $\psi(x)$ referred to as the wavelet function.

By convention we define the translates of $\varphi(x)$ as

$$\varphi_i(x) := \varphi_0(x-i) := \varphi(x-i)\ . \tag{1}$$

The scaling relation that defines $\varphi(x)$ is

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x). \tag{2}$$

The scaling relation that defines $\psi(x)$ is

$$\psi(x) = \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x+k). \tag{3}$$

From the scaling relation (2) above one sees that $\varphi(x)$ is equal to a sum of scaled and shifted versions of itself. The scaling factor in this equation is 2 and hence we refer to $\varphi(x)$ and $\psi(x)$ of this form as a *multiplier* 2 system. One can generalize wavelet systems to an arbitrary nonnegative integer multiplier. For higher order multipliers there are multiple functions $\psi(x)$ with different sets of a_k coefficients. For the purposes of this paper, however, we restrict ourselves to the multiplier 2 case with real valued a_k's.

The wavelet expansion of a function $f : \mathbf{R} \to \mathbf{C}$ is of the form

$$f(x) = \sum_{l \in \mathbf{Z}} c_l \varphi_l(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbf{Z}} c_{jk} \psi_{jk}(x)\ .$$

The indices k and j represent translation and scaling respectively.

$$\varphi_{jl}(x) := 2^{j/2} \varphi(2^j x - l)\ \ ,\ \ \varphi_l(x) := \varphi_{0l}(x) \tag{4}$$

$$\psi_{jk}(x) := 2^{j/2} \psi(2^j x - k) \tag{5}$$

If $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only.

$$f(x) = \sum_{l \in \mathbf{Z}} c_{Jl} \varphi_{Jl}(x) \tag{6}$$

This simple but important relation can be found by repeated application of the scaling relation (see [6]). Therefore a finite wavelet expansion of $f(x)$ can be written solely in terms of translated scaling functions. For ease of notation we remove the J from our equations and assume that the $\varphi(x)$'s are all at the finest level of resolution.

The scaling functions we have used are those of Daubechies [2]. In her work, Daubechies found and exploited the link between vanishing moments of the wavelet $\psi(x)$ and regularity of the wavelet $\psi(x)$ and scaling function $\varphi(x)$. We say that the wavelet $\psi(x)$ has K vanishing moments if

$$\int x^k \psi(x) dx = 0 \text{ for } 0 \leq k \leq K \; .$$

A necessary and sufficient condition for this to hold is that integer translates of the scaling function $\varphi(x)$ perfectly interpolate polynomials of degree up to K; that is, for each k, $0 \leq k \leq K$ there exist constants c_l such that

$$x^k = \sum c_l \varphi_l(x) \; .$$

Daubechies [2] introduced scaling functions satisfying this property and distinguished by having the shortest possible support. The scaling function DN (where N is an even integer) will have support $[0, N-1]$ and $(N/2-1)$ vanishing wavelet moments. In [2] and with a refined analysis in [3] Daubechies showed that there exists $\lambda > 0$ such that DN has $\lambda(N/2-1)$ continuous derivatives; for small N, $\lambda \geq 0.55$.

2.2 Derivatives and Connection Coefficients

In order to solve PDE's we will need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. We first define the following shorthand for differentiation of a function:

$$\varphi_l^n := \frac{d^n \varphi_l(x)}{dx^n} \tag{7}$$

In general a superscript will represent differentiation. From equation (6) we derive the Galerkin approximation of a derivative of $f(x)$ as

$$f^d(x) = \sum_l c_l \varphi_l^d(x) \; . \tag{8}$$

Now $\varphi_l^d(x)$ can be approximated in terms of $\varphi(x)$ as,

$$\varphi_l^d(x) = \sum_m \lambda_m \varphi_m(x) \tag{9}$$

where

$$\lambda_m = \int_{-\infty}^{\infty} \varphi_l^d(x) \varphi_m(x) dx. \tag{10}$$

The coefficient λ_m is a 2-term connection coefficient which in its most general form is defined as

$$\Lambda_{l_1 l_2}^{d_1 d_2} := \int_{-\infty}^{\infty} \varphi_{l_1}^{d_1}(x) \varphi_{l_2}^{d_2}(x) dx. \tag{11}$$

There is an analogous connection coefficient for the function $\psi(x)$ but by use of equation (6) there is no necessity for calculating these values. It is, however, possible to derive a connection coefficient which is an integral of products of differentiated $\varphi(x)$'s and $\psi(x)$'s from just the scaling function connection coefficients (demonstrated below). For the rest of this paper the reference to connection coefficient will always imply scaling function connection coefficient.

In general, a connection coefficient is a coefficient in the Galerkin expansion (approximation) of a function of the form

$$f(x) := \prod_{i=1}^{n} \varphi_{l_i}^{d_i} \ . \tag{12}$$

The resulting connection coefficient is

$$\int_{-\infty}^{\infty} f(x)\varphi_l(x)dx \ . \tag{13}$$

In the most general case we allow φ_l to be differentiated which gives rise to the n-term connection coefficient:

$$\Lambda(l_1, l_2, \ldots, l_n, d_1, d_2, \ldots, d_n) := \Lambda_{l_1 l_2 \ldots l_n}^{d_1 d_2 \ldots d_n} := \int_{-\infty}^{\infty} \prod_{i=1}^{n} \varphi_{l_i}^{d_i}(x)\, dx \tag{14}$$

3 Outline of the General Method

3.1 Statement of Problem

In this paper we will show how to evaluate integrals from equation (14) for $n = 2, 3$. Using the change change of variables $u = x - l$, we can alter a doubly subscripted connection coefficient into a singly subscripted one, and a triply subscripted connection coefficient into a doubly subscripted one. This transformation is useful because it simplifies the generation of the linear system of equations. We therefore define the two and three term connection coefficients as

$$\Lambda_l^{d_1 d_2} := \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_l^{d_2}(x)dx \tag{15}$$

and

$$\Lambda_{lm}^{d_1 d_2 d_3} := \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_l^{d_2}(x)\varphi_m^{d_3}(x)dx, \tag{16}$$

where $d_i \geq 0$.

Our method is a general technique for evaluating integrals of products of arbitrary derivatives of the scaling function. Later papers will describe how to solve the n-term connection coefficient problem. In this paper we focus on the specific case of 2-term and 3-term connection coefficients.[1] The method for evaluating the 3-term case will be presented in full and the appropriate equations for the 2-term case will be provided in appendix B.

3.2 Consequences of the Scaling Relation and Moment Properties

Let us assume that the scaling function can be differentiated as many times as required for the following operations to be performed. Successive differentiation of the scaling relation and substitution for the integrand in equation (14) yields a system of homogeneous linear equations that the connection coefficients must satisfy. It can be shown that the solution space (eigenspace) of the homogeneous linear equations has some positive multiplicity, which depends on the number of terms in the connection coefficients.

In order to uniquely specify a solution (connection coefficient) we develop a set of linearly independent equations equal to the dimension of the solution space. One condition will be a inhomogeneous equation that fixes the normalization. We expand the monomial x^k using scaling functions and wavelets, and take the inner product of both sides with the scaling function $\varphi(x)$. This yields an equation relating a moment of $\varphi(x)$ to the connection coefficients. We repeat this process with different values of k until the solution is uniquely specified.

4 3-Term Connection Coefficients

4.1 The Scaling Equations

When N in equation (2) is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0, N-1]$. For a fixed triple (d_1, d_2, d_3), only finitely many of the $\Lambda_{lm}^{d_1 d_2 d_3}$ are nonzero, namely those for which

$$2-N \leq l \leq N-2, \quad 2-N \leq m \leq N-2, \quad \text{and} \quad |l-m| \leq N-2.$$

There are $M = 3N^2 - 9N + 7$ such pairs (l, m). We shall denote this set of M pairs (l, m) by S. Select a fixed but arbitrary bijection of the set S onto the integers $\{1, \ldots, M\}$ and let $\Lambda^{d_1 d_2 d_3}$ be an M-vector whose components are the numbers $\Lambda_{lm}^{d_1 d_2 d_3}$.

Our strategy is to solve for the vector $\Lambda^{d_1 d_2 d_3}$ by creating a system of M or more linear equations that have the $\Lambda_{lm}^{d_1 d_2 d_3}$ as their unique solution. We assume that the triple (d_1, d_2, d_3) is fixed, and we derive two classes of linear and affine

[1] 2-term connection coefficients have also been studied by Beylkin [1]. These 2-term integrands were introduced at Aware, Inc. by H. L. Resnikoff and documented in Chapter 6 of *Introduction to Arithmeticum Analysis*, 26 March 1988, an Aware Technical Report. See [4] for further analysis of these functionals.

relations for the vector $\Lambda^{d_1 d_2 d_3}$. Let us assume that $\varphi(x)$ is d-times differentiable. Differentiating the scaling relation d times, we obtain

$$\varphi^d(x) = 2^d \sum_{k=0}^{N-1} a_k \varphi_k^d(2x).$$

Substituting the right side of the scaling relation for $\varphi(x)$ into the definition of the 3-term connection coefficient $\Lambda_{lm}^{d_1 d_2 d_3}$ yields

$$\begin{aligned}
\Lambda_{lm}^{d_1 d_2 d_3} &= 2^{d_1+d_2+d_3} \int_{-\infty}^{\infty} \sum_p a_p \varphi_p^{d_1}(2x) \sum_q a_q \varphi_{2l+q}^{d_2}(2x) \sum_r a_r \varphi_{2m+r}^{d_3}(2x) dx \\
&= 2^{d_1+d_2+d_3} \sum_{p,q,r} a_p a_q a_r \int_{-\infty}^{\infty} \varphi_p^{d_1}(2x)\, \varphi_{2l+q}^{d_2}(2x)\, \varphi_{2m+r}^{d_3}(2x)\ dx \\
&= 2^{d_1+d_2+d_3-1} \sum_{p,q,r} a_p a_{q-2l+p} a_{r-2m+p} \int_{-\infty}^{\infty} \varphi^{d_1}(x)\, \varphi_q^{d_2}(x)\, \varphi_r^{d_3}(x)\ dx.
\end{aligned}$$

After simplification we find

Theorem 1

$$A\Lambda^{d_1 d_2 d_3} = \frac{1}{2^{d-1}} \Lambda^{d_1 d_2 d_3}$$

where

$$d := d_1 + d_2 + d_3$$

and

$$A_{l,m:q,r} := \sum_p a_p a_{q-2l+p} a_{r-2m+p}\ .$$

We refer to these equations as the *Scaling Equations*.

It is clear that the scaling equations cannot have a unique nonzero solution since they are homogeneous. The set of vectors which satisfy the scaling equations form a vector space of nonzero dimension. By construction, the connection coefficients are solutions of the system and are in this vector space. We examine the number of linearly independent connection coefficients in this space and later show how to uniquely specify a particular connection coefficient.

Since the coefficients in the scaling equations only depend on $d = d_1 + d_2 + d_3$, and not otherwise on the individual derivatives d_1, d_2, and d_3, the vector $\Lambda^{c_1 c_2 c_3}$ will be a solution of the linear system of scaling equations whenever $c_1 + c_2 + c_3 = d$.

The vectors $\Lambda^{c_1 c_2 c_3}$ are not all linearly independent. For all l and m, the function $\varphi^{c_1 - 1}(x)\varphi_l^{c_2}(x)\varphi_m^{c_3}(x)$ has compact support, so we have the relation

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} \left(\varphi^{c_1-1}(x)\varphi_l^{c_2}(x)\varphi_m^{c_3}(x)\right)' dx \\
&= \int_{-\infty}^{\infty} \big(\varphi^{c_1}(x)\varphi_l^{c_2}(x)\varphi_m^{c_3}(x) \\
&\qquad + \varphi^{c_1-1}(x)\varphi_l^{c_2+1}(x)\varphi_m^{c_3}(x) + \varphi^{c_1-1}(x)\varphi_l^{c_2}(x)\varphi_m^{c_3+1}(x)\big)\, dx
\end{aligned}$$

whence

Lemma 1

$$\Lambda^{c_1,c_2,c_3} + \Lambda^{c_1-1,c_2+1,c_3} + \Lambda^{c_1-1,c_2,c_3+1} = 0.$$

This allows us to express the vector $\Lambda^{c_1c_2c_3}$ as a linear combination of other vectors $\Lambda^{b_1b_2b_3}$ with $b_1 < c_1$ and $b_1 + b_2 + b_3 = c_1 + c_2 + c_3 = d_1 + d_2 + d_3 = d$. Repeating this process, we see that the vector $\Lambda^{c_1c_2c_3}$ is a linear combination of vectors of the form Λ^{0,b_2,b_3}. In fact, we can see this explicitly, via the formula

Lemma 2

$$\Lambda^{c_1c_2c_3} = (-1)^{c_1} \sum_{i=0}^{c_1} \binom{c_1}{i} \Lambda^{0,c_2+i,c_3+c_1-i}.$$

We show in section 4.3 that when the the set of vectors $\{\Lambda^{0,b,d-b} \mid b = 0, 1, \ldots, d\}$ exist, they are linearly independent. Intermediate results will not depend on this fact, and so without circular logic we have

Theorem 2 *The space spanned by the vectors*

$$\left\{ \Lambda^{c_1c_2c_3} \,\middle|\, c_1 + c_2 + c_3 = d \right\}$$

is spanned by the $(d+1)$ *linearly independent vectors*

$$\{ \Lambda^{0,b,d-b} \mid b = 0, 1, \ldots, d \}.$$

In order to uniquely determine the expressions $\Lambda^{0,b,d-b}$, we need to find at least $(d+1)$ additional independent conditions or equations which are still independent when restricted to the $(d+1)$-dimensional space spanned by the $\Lambda^{0,b,d-b}$. We will obtain d independent homogeneous linear equations and one inhomogeneous linear normalization equation from equations involving the moments of $\varphi(x)$. These will complete the system of equations and lead to a unique determination of the three term connection coefficients when the dimension of the solution space is $d+1$.

This point is examined further in section 5.2. For the purposes of this paper it is assumed that the solution space has exact dimension $d+1$. In practice, this can be verified by construction.

4.2 Moment Equations

Let K be the largest integer k such that x^k can be represented exactly by translations of the scaling function. That is, if $0 \le k \le K$, then there exist c_l such that

$$x^k = \sum c_l \varphi_l(x) \; . \qquad (17)$$

The expansion coefficients c_l will be given by

$$c_l = M_l^k := \mathrm{Mom}_k(\varphi_l(x)) = \int_{-\infty}^{\infty} x^k \varphi_l(x) dx \ . \tag{18}$$

It follows from Theorem 2 that the dimension of the solution space of the linear system of scaling equations is at least $(d+1)$ when $d \leq K$. We conjecture that the dimension of the solution space is exactly of dimension $(d+1)$ when $d \leq K$; numerical experiments support this conjecture. Techniques for calculating the moments are described in Appendix A.

Differentiating the scaling function expansion (17) k times, we obtain

$$k! = \sum_i M_i^k \varphi_i^k(x).$$

Further differentiation yields

$$0 = \sum_i M_i^k \varphi_i^j(x) \quad \text{whenever} \quad j > k.$$

Multiplying both sides of the same equation by $\varphi^{d_1}(x)\varphi_m^{d_3}(x)$ and integrating over $\mathbf{R}$, we obtain the relation

Theorem 3
$$\sum_l M_l^k \Lambda_{lm}^{d_1 d_2 d_3} = \begin{cases} k! \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_m^{d_3}(x)\, dx & \textit{for } k = d_2, \\ 0 & \textit{for } k < d_2. \end{cases} \tag{19}$$

Similarly, multiplying both sides of this equation by $\varphi^{d_1}(x)\varphi_l^{d_2}(x)$ and integrating over $\mathbf{R}$, we obtain the relation

Theorem 4
$$\sum_m M_m^k \Lambda_{lm}^{d_1 d_2 d_3} = \begin{cases} k! \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_l^{d_2}(x)\, dx & \textit{for } k = d_3, \\ 0 & \textit{for } k < d_3. \end{cases} \tag{20}$$

We refer to the equations in theorems 3 and 4 as the *Moment Equations*.

4.3 Reduction by Moment Equations

We now show how to generate additional equations from the moment equations such that the new equations, the scaling equations, and the moment equations will all be linearly independent. By Lemma 2 we may assume $d_1 = 0$. Now

$$\sum_l M_l^0 \Lambda_{lm}^{0,0,d_3} = \int_{-\infty}^{\infty} \varphi(x)\varphi_m^{d_3}(x) dx \ , \tag{21}$$

and if we choose m such that

$$\int_{-\infty}^{\infty} \varphi(x)\varphi_m^{d_3}(x)\, dx \neq 0 \ ,$$

then the homogeneous moment equation

$$\sum_l M_l^0 \Lambda_{lm}^{0,d_2,d_3} = 0 \tag{22}$$

is not satisfied by $\Lambda^{0,0,d}$, though it is satisfied by $\Lambda^{0,b,d-b}$ for $b = 1, 2, \ldots, d$. The choice of m is determined by the fact that the values of $\int_{-\infty}^{\infty} \varphi(x)\varphi_m^{d_3}(x)\,dx$ are symmetric when d_3 is even and antisymmetric when d_3 is odd. The integral attains its maximum value when $m = 0$ (d_3 even), and $m = 1$ (d_3 odd), so it is numerically advantageous to choose this value of m to make the computation better posed.

After adjoining the moment equation (21) to the scaling equations the solution space of the system is spanned by the vectors

$$\{ \Lambda^{0,b,d-b} \mid b = 1, 2, \ldots, d \},$$

and hence has dimension d. While $\Lambda^{0,0,d}$ is a solution to the homogeneous scaling equations, it yields a nonzero right hand side for the equation (21). Thus the moment equation (21) is linearly independent with respect to the scaling equation.

Similarly, the homogeneous moment equation

$$\sum_l M_l^1 \Lambda_{lm}^{0,d_2,d_3} = 0$$

is satisfied by $\Lambda^{0,b,d-b}$ for any $b > 1$, but not by $\Lambda^{0,1,d-1}$. We do not know whether this equation is satisfied by $\Lambda^{0,0,d}$, but this is unimportant, since this information is not needed to conclude that the space of vectors satisfying the moment equations as well as the scaling equations is spanned by the vectors

$$\{ \Lambda^{0,b,d-b} \mid b = 2, 3, \ldots, d \}.$$

Proceeding inductively, we see that the space of vectors satisfying the scaling equations as well as the d_2 homogeneous moment equations

$$\sum_l M_l^b \Lambda_{lm}^{0,d_2,d_3} = 0,$$

where b varies from 0 to $d_2 - 1$ inclusive, and m is as described above, is spanned by the vectors

$$\{ \Lambda^{0,b,d-b} \mid d_2 \le b \le d \}.$$

The strategy we have employed is to adjoin additional moment equations that reduce the rank of the solution space by guaranteeing that they are not a linear combination of the scaling equations and the previously adjoined moment equations. So far we have reduced the rank deficiency of the system by d_2.

Now we pick the appropriate equations to reduce it by d_3. If we also adjoin the homogeneous moment equations

$$\sum_m M_m^c \Lambda_{lm}^{0,d_2,d_3} = 0,$$

where c varies from 0 to $d_3 - 1$ inclusive and l is chosen so that

$$\int_{-\infty}^{\infty} \varphi(x) \varphi_l^{d_2}(x)\, dx \neq 0,$$

(for example, choosing $l = 0$ if d_2 is even, and $l = 1$ if d_2 is odd), then these additional equations eliminate the vectors $\Lambda^{0,b,d-b}$ with $b > d_2$ from the solution space, leaving a system with a one-dimensional solution space, all of whose solutions are scalar multiples of $\Lambda^{0,d_2,d-d_2}$.

It remains to show the second half of Theorem 2, that when the vectors $\Lambda^{0,b,d-b}$ for $b = 0, 1, \ldots, d$ exist, they are linearly independent. Let us demonstrate this now. If we assume that there exist constants c_b such that

$$\Omega_{l,m}^d = \sum_{b=0}^{d} c_b \Lambda_{lm}^{0,b,d-b} = 0$$

then

$$c_b = 0$$

for $b = 0, 1, \ldots, d$.

For instance, examine

$$\sum_l M_l^0 \Omega_{l,m}^d = \sum_{l,b} c_b M_l^0 \Lambda_{lm}^{0,b,d-b} = 0.$$

We have seen from Section 4.2 that

$$\sum_l M_l^0 \Lambda_{lm}^{0,b,d-b} \neq 0$$

when $b = 0$ and that

$$\sum_l M_l^0 \Lambda_{lm}^{0,b,d-b} = 0$$

when $b > 0$. Therefore, in order that the summation over Ω_{lm}^d to be 0 we must have $c_0 = 0$. In a similar fashion one can show that for $k \leq K$

$$\sum_l M_l^k \Omega_{l,m}^d = 0$$

implies $c_k = 0$.

The above also demonstrates that equation (19) and equation (20) are independent of the scaling relations when they are restricted to the space

$$\{\Lambda^{0,b,d-b} \mid b = 0, 1, \ldots, d\}.$$

That is, for each vector there is at least one moment equation that does not vanish when applied to the vector (is inhomogeneous).

4.4 Normalization of the Solution

To distinguish the desired solution vector $\Lambda^{0,d_2,d-d_2}$ from its scalar multiples, it suffices to find any single inhomogeneous equation satisfied by $\Lambda^{0,d_2,d-d_2}$, which, by virtue of its inhomogeneity, will not be satisfied by any scalar multiple of $\Lambda^{0,d_2,d-d_2}$. The following describes a method that uses moment equations to create a linearly independent inhomogeneous equation. Note that we will generalize the method to an n-term connection coefficient.

Assume that $d_1 = 0$, and that we seek to calculate the values of the connection coefficients $\Lambda^{0,d_2,d_3,\ldots,d_n}_{0,l_2,l_3,\ldots,l_n}$ for fixed values of n and the $\{d_i\}$. We obtain such an equation in the following way. Assuming that $\psi(x)$ has at least d_i vanishing moments, we have the equation

$$x^{d_i} = \sum_j M_j^{d_i} \varphi_j(x).$$

Differentiating d_1 times, we obtain

$$(d_i)! = \sum_j M_j^{d_i} \varphi_j^{d_i}(x).$$

Taking this equation for each value of i from 2 through n, and multiplying them together, we obtain

$$\prod_{i=2}^{n}(d_i)! = \sum_{l_2,l_3,\ldots,l_n} \left(\prod_{i=2}^{n} M_{l_i}^{d_i}\right) \varphi_{l_2}^{d_2}(x)\varphi_{l_3}^{d_3}(x)\ldots\varphi_{l_n}^{d_n}(x).$$

If we multiply both sides by $\varphi(x)$, integrate, and use the fact that $\int \varphi(x)dx = 1$, we obtain

$$\prod_{i=2}^{n}(d_i)! = \sum_{l_2,l_3,\ldots,l_n} \left(\prod_{i=2}^{n} M_{l_i}^{d_i}\right) \Lambda^{0,d_2,d_3,\ldots,d_n}_{0,l_2,l_3,\ldots,l_n}.$$

The summation is extended over all values of the $\{l_i\}$ such that $(0, l_2, l_3 \ldots l_n) \in S$. All other values of the l_i can be neglected, since they make no contribution to the value of the integral. The specific equation for the 3-term connection coefficient case is

$$d_2!d_3! = \sum_{lm} M_l^{d_2} M_m^{d_3} \Lambda^{0,d_2,d_3}_{lm}.$$

The addition of this equation to the scaling equations and the d homogeneous moment equations described above yields a linear system with the desired unique solution $\Lambda^{0,d_2,d-d_2}$. There are, however, other inhomogeneous equation that can be used for this purpose. They are somewhat less desirable for reasons that will be explained below, but we describe them here for completeness.

Recall that for any scaling function $\varphi(x)$, we have the formula

$$\sum_{i=-\infty}^{\infty} \varphi_i(x) = 1.$$

This follows directly from the vanishing of the zeroth moment of the associated wavelet and the orthonormality of $\varphi(x)$ and its integer translates. Choosing l and m so that

$$\int_{-\infty}^{\infty} \varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx \neq 0,$$

(for example, by choosing $l = m = 0$ if $d_2 + d_3$ is even, and $l = 0$ and $m = 1$ if $d_2 + d_3$ is odd), we have the inhomogeneous equation

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx &= \int_{-\infty}^{\infty} \left(\sum_{i=-\infty}^{\infty} \varphi_i(x)\right) \varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx \\
&= \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_i(x)\varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx \\
&= \sum_{i=\max(l,m)-N+2}^{\min(l,m)+N-2} \int_{-\infty}^{\infty} \varphi(x)\varphi_{l-i}^{d_2}(x)\varphi_{m-i}^{d_3}(x)\,dx \\
&= \sum_{i=\max(l,m)-N+2}^{\min(l,m)+N-2} \Lambda_{l-i,m-i}^{0,d_2,d_3}\,. \qquad (23)
\end{aligned}$$

There are two major advantages to be found in using the inhomogeneous moment equation. First, the method described above requires knowledge of the value of an $(n-1)$-term connection coefficient. Therefore we are required to recursively compute the n-term connection coefficients as a function of $(n-1)$-term connection coefficients; this could be computationally expensive. Use of the inhomogeneous moment equation allows the calculation of n-term connection coefficients directly, and computing the moments of $\varphi(x)$ and its translates is computationally very easy (see Appendix A).

The second advantage of the inhomogeneous moment equation is in the calculation of $\Lambda_{0,l_2,l_3,\ldots,l_n}^{0,d_2,d_3\ldots d_n}$ for a function $\varphi(x)$ which is not actually $\max_i(d_i)$ times differentiable. In some, but not all, cases, this is possible, as long as $\varphi(x)$ is formally $\max_i(d_i)$ times differentiable, and the integrand is understood in the sense of distributions. The inhomogeneous connection coefficient equation (23) reduces the computation of an n-term scaling integral to the computation of an $(n-1)$-term connection coefficient with the same value of $\max_i(d_i)$. In the case where $\varphi(x)$ is actually $\max_i(d_i)$ times differentiable, this does not present a problem. However, in the case where $\varphi(x)$ is merely formally $\max_i(d_i)$ times differentiable, it is possible that the $(n-1)$-term connection coefficient may not exist. Since it does not rely on the calculation of any integrals, the moment technique avoids this problem.

5 Solutions, Existence and Other Calculations

5.1 Solving the Resulting System

We have created a system of $M+d+1$ equations in M unknowns. It has rank M, and therefore a unique solution. The simplest way to solve our system of equations is to choose a subset of M of the equations that has a unique solution, and then solve the resulting square system for example, by means of the LU decomposition. Elementary linear algebra guarantees that such a subsystem exists. Since the scaling equations themselves form a system with rank $M-(d+1)$, any such subsystem must clearly include all d moment equations, as well as the inhomogeneous equation. We conjecture that a subsystem formed from the d moment equations, the inhomogeneous equation, and any $M-(d+1)$ of the scaling equations has full rank. The construction of solutions in all specific instances supports this conjecture.

One can ensure that the system chosen has full rank, rather than being an anomalous rank-deficient system that appeared to be of full rank due to numerical error, by verifying that the $d+1$ unused scaling equations are also satisfied. If numerical difficulties are encountered, the system of $M+d+1$ equations in M unknowns can be solved using any of the standard techniques for solving over-determined systems, such as the QR algorithm.

5.2 Domain of Applicability

Under the conditions that $d \leq K$, and the dimension of the solution space of the scaling equations is exactly $d+1$, our construction for evaluating connection coefficients is well defined, and it will produce a definite solution. In the case where the order of differentiation is less than the classical differentiability of the scaling function the algorithm must produce the unique result that would be obtained by Riemann integration. When the order of differentiation is greater than the classical order, but the scaling function is *formally* differentiable, our construction defines a consistent extension of the definition of the connection coefficient functional which will be discussed elsewhere.

It is important, however, to check that the algorithm does not require that the wavelet $\psi(x)$ associated with $\varphi(x)$ have more vanishing moments than are implied by the formal differentiability required of $\varphi(x)$. Calculation of the integrals

$$\Lambda_{lm}^{0,d_2,d_3} = \int_{-\infty}^{\infty} \varphi(x)\varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx,$$

using the moment equations requires that $\max(d_2, d_3) \leq K$. If we abide by the stronger restriction that $d \leq K$ then we know that $\psi(x)$ has at least $d_2 + d_3$ vanishing moments so there is no question about the existence of the moment equations.

The above analysis presents an interesting question. What happens when $d > K$ and $\max(d_2, d_3) \leq K$? In this case, not all the $\Lambda^{0,b,d-b}$ may exist. In effect, however, we can still think of the $\Lambda^{0,b,d-b}$ as a basis of the solution space of scaling equations. We only solve for those $\Lambda^{0,b,d-b}$ that formally do exist.

Numerically we have found that the dimension of the solution space is $(d+1)$ when $d \leq N-1$. For instance, this is valid for D6, D8 and D10 (D4 seems to form an exception to this rule that requires further examination). At the present time we have no explanation of this fact. For larger values of N and d the numerical results can be complicated, by the occurrence of spurious eigenvalues numerically close to 2^{1-d}. This can cause the matrices to be ill-conditioned. For $d \leq K$ this problem does not appear to arise.

In the cases where $\varphi(x)$ is formally, but not actually, $\max(d_2, d_3)$ times differentiable, the factors in the integrand must be understood in the sense of distributions. The above algorithms provide a technique for calculating the desired integrals of these distributions if they converge.

In the case where we wish to calculate integrals of the form

$$\Lambda_{lm}^{d_1 d_2 d_3} = \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_l^{d_2}(x)\varphi_m^{d_3}(x)\,dx,$$

we do so using the formula

$$\Lambda_{lm}^{d_1 d_2 d_3} = (-1)^{d_1} \sum_{i=0}^{d_1} \binom{d_1}{i} \Lambda_{lm}^{0,d_2+i,d_3+d_1-i}$$

which was derived by repeated integration by parts. Thus this technique can be used to calculate $\Lambda_{lm}^{d_1 d_2 d_3}$ only if $\psi(x)$ has at least $d_1 + \max(d_2, d_3)$ vanishing moments. It is known that if $\varphi(x)$ is a Daubechies wavelet that is actually $\max(d_1, d_2, d_3)$ times differentiable, then the associated wavelet $\psi(x)$ must have at least $2\max(d_1, d_2, d_3)$ vanishing moments. Since this quantity is always greater than or equal to $d_1 + \max(d_2, d_3)$, this shows that the above techniques are always sufficient to calculate $\Lambda_{lm}^{d_1 d_2 d_3}$ whenever the derivatives involved in the above integral actually exist, and can also be used in some of the cases where the derivatives involved only exist in the formal sense. If it is desired to calculate $\Lambda_{lm}^{d_1 d_2 d_3}$ in a case where $d_1 + d_2$ is greater than the number of vanishing moments of $\psi(x)$, but both $d_1 + d_3$ and $d_2 + d_3$ are less than this number, this can still be done (the terms in the integrand again being understood in the sense of distributions) with the above techniques by a change of variables, using the formula

$$\Lambda_{lm}^{d_1 d_2 d_3} = \Lambda_{l-m,-m}^{d_3 d_2 d_1}.$$

Similarly, if both $d_1 + d_2$ and $d_2 + d_3$ are less than the number of vanishing moments of $\psi(x)$, but $d_1 + d_3$ is greater, the desired integrals can be calculated formally with the above techniques after applying the transformation

$$\Lambda_{lm}^{d_1 d_2 d_3} = \Lambda_{-l,m-l}^{d_2 d_1 d_3}.$$

5.3 Dilations and Wavelets

It is important to note that once we have a technique for calculating the $\Lambda_{lm}^{d_1 d_2 d_3}$, we also have a technique for calculating many other important integrals. Since integration, differentiation, and translation are all linear operators, we can calculate the integral

$$\int_{-\infty}^{\infty} \alpha_k^{d_1}(x) \beta_l^{d_2}(x) \gamma_m^{d_3}(x)\, dx$$

whenever α, β, and γ are known linear combinations of $\varphi(x)$ and its translates. In particular, this lets us calculate the exact values of integrals of products of three terms, where each term is a derivative of a translate of any of $\varphi(x)$, $\psi(x)$, a dilation by a power of two of $\varphi(x)$, or a dilation by a power of two of $\psi(x)$ by using equations (2), (3), (4) and (5).

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A Calculation of Moments

In order to calculate the values of the scaling integrals $\Lambda^{d_1 d_2 \ldots d_n}_{l_1 l_2 \ldots l_n}$, it is first necessary to calculate the values of the moments of $\varphi(x)$ and its translates,

$$M_i^k = \int_{-\infty}^{\infty} x^k \varphi(x-i)\, dx.$$

The techniques used to calculate these moments are a simpler version of the techniques used to calculate the scaling integrals themselves. In both cases, the techniques rely on the scaling relation to derive sufficient linear and affine relations among the quantities being calculated to determine them uniquely. This allows the rapid calculation of the desired quantities, as well as showing that they are rational functions of the scaling coefficients.

By definition, the scaling function $\varphi(x)$ is normalized so that $M_0^0 = 1$, and a change of variables shows that $M_i^0 = 0$ for all i. We calculate the moments M_i^k by induction on k, so we assume that M_i^j has already been calculated for all i for all $j < k$.

The scaling relation yields the formula

$$M_0^j = \int_{-\infty}^{\infty} x^j \varphi(x)\, dx = \int_{-\infty}^{\infty} x^j \sum_{i=0}^{N-1} a_i \varphi(2x-i)\, dx$$

$$= 2^{-j-1} \sum_{i=0}^{N-1} a_i \int_{-\infty}^{\infty} (2x)^j \varphi(2x-i))\, d(2x) = 2^{-j-1} \left(\sum_{i=0}^{N-1} a_i M_i^j \right).$$

To reduce the number of unknowns in this equation, we will eliminate M_i^j for

$i \neq 0$. The substitution $u = x - i$ implies

$$\begin{aligned} M_i^j &= \textstyle\int_{-\infty}^{\infty} x^j \varphi(x-i)\,dx \\ &= \textstyle\int_{-\infty}^{\infty} (u+i)^j \varphi(u)\,du \\ &= \textstyle\sum_{k=0}^{j} \binom{j}{k} i^{j-k} \int_{-\infty}^{\infty} u^k \varphi(u)\,du \\ &= \textstyle\sum_{k=0}^{j} \binom{j}{k} i^{j-k} M_0^k. \end{aligned}$$

Substituting this into the previous equation, we obtain

$$\begin{aligned} M_0^j &= 2^{-j-1}\left(\textstyle\sum_{i=0}^{N-1} a_i M_i^j\right) \\ &= 2^{-j-1}\left(\textstyle\sum_{i=0}^{N-1} a_i \sum_{k=0}^{j} \binom{j}{k} i^{j-k} M_0^k\right) \\ &= 2^{-j-1}\textstyle\sum_{k=0}^{j-1} \binom{j}{k} M_0^k \left(\sum_{i=0}^{N-1} a_i i^{j-k}\right) + 2^{-j-1} M_0^j \left(\sum_{i=0}^{N-1} a_i\right) \\ &= 2^{-j-1}\textstyle\sum_{k=0}^{j-1} \binom{j}{k} M_0^k \left(\sum_{i=0}^{N-1} a_i i^{j-k}\right) + 2^{-j} M_0^j. \end{aligned}$$

Solving for M_0^j, we find the formula

$$M_0^j = \frac{1}{2(2^j-1)} \sum_{k=0}^{j-1} \binom{j}{k} M_0^k \left(\sum_{i=0}^{N-1} a_i i^{j-k}\right).$$

Now that the jth moment of $\varphi(x)$, M_0^j, has been determined, the moments M_i^j of the translates of $\varphi(x)$ can be obtained using the formula

$$M_i^j = \sum_{k=0}^{j} \binom{j}{k} i^{j-k} M_0^k$$

derived above. This results in the explicit formula

Theorem 5

$$M_i^j = \frac{1}{2(2^j-1)} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_0^l \left(\sum_{i=0}^{N-1} a_i i^{k-l}\right).$$

We have just expressed the moments of the scaling function in terms of the "discrete" moments of the scaling coefficients.

B 2-Term Connection Coefficient Equations

In this appendix we present the equations for the construction of the 2-term connection coefficients. In general they are a simplification of the 3-term equations. For the 2-term case, however, the dimension of the solution space, when it exists, is always equal to 1 (see [4] for a discussion of existence). This can easily be seen from the fact that the integration by parts of a 2-term product produces an equation of the form

$$\Lambda_l^{d_1 d_2} = -\Lambda_l^{d_1-1,d_2+1} .$$

We therefore only need one inhomogeneous independent equation to normalize the system of equations. For this we use a simple moment equation.

From the derivation of the 3-term case one can see that

Theorem 6

$$A\Lambda^{d_1 d_2} = \frac{1}{2^{d-1}}\Lambda^{d_1 d_2}$$

where

$$d := d_1 + d_2$$

and

$$A_{l:q} := \sum_p a_p a_{q-2l+p} .$$

From integration by parts we can always transform $\Lambda^{d_1 d_2}$ into $\Lambda^{0,d}$ where $d = d_1 + d_2$ and $\Lambda^{0,d}$ exists. This allows us to use the scaling function expansion of x^k as an independent inhomogeneous equation. From the derivation of the 3-term case one can see that

Theorem 7

$$d! = (-1)^d \sum_l M_l^d \Lambda_l^{0,d} .$$

C Connection Coefficients for D6

In this appendix we present the values for the connection coefficients required to solve Burgers' equation using the D6 wavelet basis. This requires a vector of 2-term connection coefficients for the diffusion, u_{xx}, and a matrix of 3-term coefficients for the nonlinear advection, uu_x. The 2-term values are indexed over a single index l and the 3-term matrix is indexed over two indices l and m.

The D6 scaling function is defined by the scaling recursion

$$\varphi(x) = \sum_{k=0}^{5} a_k \varphi(2x-k)$$

where the coefficients a_k are the numbers in the following table.

Table 1: Scaling coefficients for D6

$$
\begin{aligned}
a_0 &= \tfrac{1+\sqrt{10}\pm\sqrt{5+2\sqrt{10}}}{16} \\
a_1 &= \tfrac{5+\sqrt{10}\pm3\sqrt{5+2\sqrt{10}}}{16} \\
a_2 &= \tfrac{10-2\sqrt{10}\pm2\sqrt{5+2\sqrt{10}}}{16} \\
a_3 &= \tfrac{10-2\sqrt{10}\mp2\sqrt{5+2\sqrt{10}}}{16} \\
a_4 &= \tfrac{5+\sqrt{10}\mp3\sqrt{5+2\sqrt{10}}}{16} \\
a_5 &= \tfrac{1+\sqrt{10}\mp\sqrt{5+2\sqrt{10}}}{16}
\end{aligned}
$$

Table 2: 2-term connection coefficients for D6

$$\Lambda[l] := \int_{-\infty}^{\infty} \varphi^{(2)}(x)\varphi_l^{(0)}(x)\,dx$$

$\Lambda[-4]$	=	$+0.00535714285714$	$\Lambda[1]$	=	$+3.39047619047638$
$\Lambda[-3]$	=	$+0.11428571428571$	$\Lambda[2]$	=	-0.87619047619052
$\Lambda[-2]$	=	-0.87619047619052	$\Lambda[3]$	=	$+0.11428571428571$
$\Lambda[-1]$	=	$+3.39047619047638$	$\Lambda[4]$	=	$+0.00535714285714$
$\Lambda[0]$	=	-5.26785714285743			

Table 3: 3-term connection coefficients for D6

$$\Lambda[l,m] := \int_{-\infty}^{\infty} \varphi^{(1)}(x)\varphi_l^{(0)}(x)\varphi_m^{(0)}(x)dx$$

$\Lambda[-4,-4]$	=	+0.00000152412730	$\Lambda[-3,-4]$	=	+0.00003290196944
$\Lambda[-4,-3]$	=	+0.00003290196944	$\Lambda[-3,-3]$	=	+0.00145229718171
$\Lambda[-4,-2]$	=	−0.00013589006821	$\Lambda[-3,-2]$	=	−0.00617732215645
$\Lambda[-4,-1]$	=	+0.00044515935738	$\Lambda[-3,-1]$	=	+0.02049168849649
$\Lambda[-4,0]$	=	−0.00000122963248	$\Lambda[-3,0]$	=	−0.00118818208361
			$\Lambda[-3,1]$	=	+0.00000048873855
$\Lambda[-2,-4]$	=	−0.00013589006821	$\Lambda[-1,-4]$	=	+0.00044515935738
$\Lambda[-2,-3]$	=	−0.00617732215645	$\Lambda[-1,-3]$	=	+0.02049168849650
$\Lambda[-2,-2]$	=	+0.03290268866112	$\Lambda[-1,-2]$	=	−0.12527254588949
$\Lambda[-2,-1]$	=	−0.12527254588946	$\Lambda[-1,-1]$	=	+0.49689180110985
$\Lambda[-2,0]$	=	−0.04677418571931	$\Lambda[-1,0]$	=	+0.31358775297548
$\Lambda[-2,1]$	=	+0.00024982636356	$\Lambda[-1,1]$	=	+0.03781026927748
$\Lambda[-2,2]$	=	+0.00000194935669	$\Lambda[-1,2]$	=	+0.00125180052383
			$\Lambda[-1,3]$	=	−0.00000044639911

$\Lambda[0,-4]$	$=$	-0.00000122963248			
$\Lambda[0,-3]$	$=$	-0.00118818208362	$\Lambda[1,-3]$	$=$	$+0.00000048873855$
$\Lambda[0,-2]$	$=$	-0.04677418571937	$\Lambda[1,-2]$	$=$	$+0.00024982636358$
$\Lambda[0,-1]$	$=$	$+0.31358775297594$	$\Lambda[1,-1]$	$=$	$+0.03781026927737$
$\Lambda[0,0]$	$=$	-0.00000000000000	$\Lambda[1,0]$	$=$	-0.24844590055475
$\Lambda[0,1]$	$=$	-0.24844590055510	$\Lambda[1,1]$	$=$	-0.62717550595142
$\Lambda[0,2]$	$=$	-0.01645134433057	$\Lambda[1,2]$	$=$	$+0.08746227661209$
$\Lambda[0,3]$	$=$	-0.00072614859085	$\Lambda[1,3]$	$=$	$+0.00492552163262$
$\Lambda[0,4]$	$=$	-0.00000076206365	$\Lambda[1,4]$	$=$	-0.00003245557033
$\Lambda[2,-2]$	$=$	$+0.00000194935669$			
$\Lambda[2,-1]$	$=$	$+0.00125180052383$	$\Lambda[3,-1]$	$=$	-0.00000044639911
$\Lambda[2,0]$	$=$	-0.01645134433055	$\Lambda[3,0]$	$=$	-0.00072614859085
$\Lambda[2,1]$	$=$	$+0.08746227661201$	$\Lambda[3,1]$	$=$	$+0.00492552163262$
$\Lambda[2,2]$	$=$	$+0.09354837143868$	$\Lambda[3,2]$	$=$	-0.02074151486006
$\Lambda[2,3]$	$=$	-0.02074151486007	$\Lambda[3,3]$	$=$	$+0.00237636416723$
$\Lambda[2,4]$	$=$	$+0.00013394071152$	$\Lambda[3,4]$	$=$	-0.00044564809593
			$\Lambda[4,0]$	$=$	-0.00000076206365
			$\Lambda[4,1]$	$=$	-0.00003245557033
			$\Lambda[4,2]$	$=$	$+0.00013394071152$
			$\Lambda[4,3]$	$=$	-0.00044564809593
			$\Lambda[4,4]$	$=$	$+0.00000245926496$