Bäcklund transformation and the Painlevé property

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Abstract

When a differential equation possesses the Painlevé property it is possible (for specific equations) to define a Bäcklund transformation (by truncating an expansion about the "singular" manifold at the constant level term). From the Bäcklund transformation, it is then possible to derive the Lax pair, modified equations and Miura transformations associated with the "completely integrable" system under consideration. In this paper, completely integrable systems are considered for which Bäcklund transformations (as defined above) may not be directly defined. These systems are of two classes. The first class consists of equations of Toda lattice type (e.g., sine-Gordon, Bullough-Dodd equations). We find that these equations can be realized as the "minus-one" equation of sequences of integrable systems. Although the "Bäcklund transformation" may or may not exist for the "minus-one" equation, it is shown, for specific sequences, that the Bäcklund transformation does exist for the "positive" equations of the sequence. This, in turn, allows the derivation of Lax pairs and the recursion operation for the entire sequence. The second class of equations consists of sequences of "Harry Dym" type. These equations have branch point singularities, and, thus, do not directly possess the Painlevé property. Yet, by a process similar to the "uniformization" of algebraic curves, their solutions may be "parametrically" represented by "meromorphic" functions. For specific systems, this is shown to provide a natural extension of the Painlevé property.

1 Introduction

Informally, the Painlevé property requires that solutions (of analytic differential equations) that arise from "good" (analytic, noncharacteristic) data be "meromorphic" (sec Refs. [1]–[10]). As described in previous works, when an equation has the Painlevé property, we may construct (auto) Bäcklund transformations by truncating an expansion of the solution about the "movable" singularity manifold at the "constant" level term.

For instance and later reference, the Korteweg-de Vries (KdV) sequence [4]

$$U_t + \frac{\partial}{\partial x} b^{n+1}(U) = 0 \ , \tag{1.1}$$

$$\frac{\partial}{\partial x} b^{n+1} = b^n_{xxx} + 2U b^n_x + U_x b^n \ , \tag{1.2}$$

for $n = 0, 1, 2, \ldots$, has the Bäcklund transformation (BT)

$$U = 4 \frac{\partial^2}{\partial x^2} \ln \phi + U_2 \ , \tag{1.3}$$

$$U_2 = -\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \ , \tag{1.4}$$

$$\frac{\phi_t}{\phi_x} + b^n(\{\phi; x\}) = 0 \ , \tag{1.5}$$

where

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{1.6}$$

is the Schwarzian derivative that is invariant under the Möbius group

$$\phi = (a\psi + b)/(c\psi + a) \ . \tag{1.7}$$

Furthermore,

$$U_3 = \{\psi; x\} \tag{1.8}$$

is a solution of Eq. (1.1) and Eq. (1.5) is invariant under the transformation

$$\phi_x = \psi_x^{-1} \ . \tag{1.9}$$

We note that

$$\begin{aligned} &b^0 = 1 \ , \qquad b^1 = U \ , \qquad b^2 = U_{xx} + \tfrac{3}{2} U^2 \ , \\ &b^3 = U_{xxxx} + 5 U U_{xx} + \tfrac{5}{2} U_x^2 + \tfrac{5}{2} U^3 \ . \end{aligned} \tag{1.10}$$

In terms of the "modified" variable

$$V = \phi_{xx}/\phi_x \ , \tag{1.11}$$

the Lenard formula (1.2) factors into

$$Db^{n+1} = (D - V)D(D + V)b^n \ , \tag{1.12}$$

where

$$\begin{aligned} D &= \frac{\partial}{\partial x} \ , \\ U &= V_x - \tfrac{1}{2}V^2 \ , \\ b^n(U) &= b^n(V_x - \tfrac{1}{2}V^2) \ . \end{aligned} \tag{1.13}$$

From Eq. (1.5) this obtains the modified KdV (MKdV) sequence

$$V_t + M_v b^n(V_x - \tfrac{1}{2}V^2) = 0 \ , \tag{1.14}$$

where

$$M_v = D(D - V) \ . \tag{1.15}$$

The KdV/MKdV sequence consists of scalar equations, local in D and first-order in $\partial/\partial t$ (evolution equation). Excepting equations that can be directly transformed into linear systems (i.e., Burgers sequence), the results of this paper argue that a scalar evolution equation with the Painlevé where property belongs to either the KdV/MKdV sequence or to the Caudrey-Dodd-Gibbon (CDG)/MCDG sequence.

The CDG sequence is the double sequence [4]

$$U_t + \theta_1 G_n(U) = 0 \ , \tag{1.16}$$

$$A_t + \theta_2 H_n(A) = 0 \ , \tag{1.17}$$

where

$$\begin{aligned} G_{n+2} &= J_1(U)\theta_1(U)G_n \ , \\ H_{n+2} &= J_2(A)\theta_2(A)H_n \ , \end{aligned} \tag{1.18}$$

$$\begin{aligned} &G_0 = 1 \ , \qquad H_0 = 1 \ , \\ &G_1 = U_{xx} + \tfrac{1}{4}U^2 \ , \qquad H_1 = A_{xx} + 4A^2 \ . \end{aligned} \tag{1.19}$$

In terms of the "modified" variables

$$U = W_x - \tfrac{1}{2}W^2 \ , \qquad A = V_x - \tfrac{1}{2}V^2 \ , \tag{1.20}$$

$$W = \psi_{xx}/\psi_x \ , \qquad V = \phi_{xx}/\phi_x \ , \tag{1.21}$$

$$\begin{aligned} &\theta_1 = (D - W)D(D + W) \ , \\ &J_1 = D^{-1}\left\{\left(D - \frac{W}{2}\right)\left(D + \frac{W}{2}\right)D\left(D - \frac{W}{2}\right)\left(D + \frac{W}{2}\right)\right\}D^{-1} \ , \end{aligned} \tag{1.22}$$

$$
\begin{aligned}
\theta_2 &= (D-V)D(D+V) \ , \\
J_2 &= D^{-1}\{(D-2V)(D-V)D(D+V)(D+2V)\}D^{-1} \ .
\end{aligned} \tag{1.23}
$$

The sequence (1.16), (1.17) has the Bäcklund transformation

$$
\begin{aligned}
U &= 12\frac{\partial^2}{\partial x^2}\ln\phi + U_2 \ , \\
A &= \frac{3}{2}\frac{\partial^2}{\partial x^2}\ln\psi + A_2 \ ,
\end{aligned} \tag{1.24}
$$

where

$$
\begin{aligned}
U_2 &= -2(\psi_{xxx}/\psi_x) \ , \\
A_2 &= -\frac{1}{2}\left(\frac{\psi_{xxx}}{\psi_x} - \frac{3}{4}\frac{\psi_{xx}^2}{\psi_x^2}\right) \ ,
\end{aligned} \tag{1.25}
$$

$$
U_3 = \{\psi; z\} \ , \qquad A_3 = \{\phi; x\} \tag{1.26}
$$

are solutions of Eqs. (1.16) and (1.17) and

$$
\begin{aligned}
\phi_t/\phi_x + H_n(\{\phi; x\}) &= 0 \ , \\
\psi_t/\psi_x + G_n(\{\psi; x\}) &= 0 \ ,
\end{aligned} \tag{1.27}
$$

possess the symmetry

$$
\psi_x = \phi_x^{-2} \ . \tag{1.28}
$$

The "modified" sequence is

$$
\begin{aligned}
W_t + M_w G_n(W_x - \tfrac{1}{2}W^2) &= 0 \ , \\
V_t + M_v H_n(V_x - \tfrac{1}{2}V^2 &= 0 \ ,
\end{aligned} \tag{1.29}
$$

where

$$
M_w = D(D-W) \ , \qquad M_v = D(D-V) \ .
$$

In term of the preceding (and somewhat more) we show in Sec. II that the sine-Gordon equation, which has a Bäcklund transformation, is the minus-one equation of the KdV sequence, while the Bullough-Dodd equation, which is shown not to have a Bäcklund transformation, is the minus-one equation of the CDG sequence. The minus-one equation of the Boussinesq sequence is shown to be equivalent to the equation for the two-dimensional, three-component periodic Toda lattice (of which the Bullough-Dodd equation is a scalar reduction). The minus-one equation of the Hirota-Satsuma sequence is also found to be an equation of Toda lattice type. In view of these results, we propose a method for constructing recursion operators of equation sequences from the annihilators of "minus-one" functionals.

In Sec. III various sequences of "Harry Dym" type equations, which have highly branched (non-Painlevé) singularities, are shown to have a "uniformization" in terms of the KdV sequence. Besides allowing the implicit definition of Bäcklund transformations, this procedure ("uniformization" of integrable systems) provides a natural extension of the Painlevé property. We note various connections with the classical theory of uniformization of algebraic curves (i.e., Schwarzian derivatives, Lax pairs).

In the appendices we present results not directly related to the discussion of Secs. II and III.

In Appendix A we find the Lax pair for the Caudrey-Dodd-Gibbon equation directly from the Bäcklund transformation and without the previously found resummation of terms. Surprisingly, the two methods for finding the Lax pair are nonequivalent.

In Appendix B we consider the factorizations of scalar, linear operators depending on one dependent variable and how this relates to the scalar reductions of the two-dimensional Toda lattice equations.

Finally, we note that the "Caudrey-Dodd-Gibbon" equation [11] has also been studied by Sawada and Kotera [12] (e.g., "Sawada-Kotera" or "Caudrey-Dodd-Gibbon-Kotera-Sawada" equation). Also, the first nontrivial integrals of the "Bullough-Dodd" equation were found in [13]. In [14] and [15] this equation was then shown to be completely integrable.

2 Minus-one functionals and the two-dimensional Toda lattice

To begin, consider the sine-Gordon equation

$$U_{xt} = \sin(U) \ , \tag{2.1}$$

which is equivalent to the equation

$$VV_{xt} - V_x V_t = \tfrac{1}{2}(V^3 - V) \ , \tag{2.2}$$

where

$$V = e^{iu} \ . \tag{2.3}$$

Equation (2.2) has the Painlevé property [1] and a Bäcklund transformation [5]

$$V = -4\frac{\partial^2}{\partial x \partial t} \ln \phi + V_2 \ , \tag{2.4}$$

where

$$V_2 = \phi_{xt}^2/\phi_x\phi_t \ , \tag{2.5}$$

$$\begin{aligned} \Omega_1 &= \{\phi; t\} + 2Z_{tt}/Z = \alpha \ , \\ \Omega_2 &= \{\phi; x\} + 2W_{xx}/W = \beta \ , \end{aligned} \tag{2.6}$$

$$\alpha\beta = \tfrac{1}{4} \ .$$

On the other hand, the Bullough-Dodd equation [13]–[15]

$$U_{xt} = ae^u - be^{-2u} \tag{2.7}$$

is equivalent to the equation

$$VV_{xt} - V_x V_t = -aV + bV^4 \ , \tag{2.8}$$

where $V = e^{-u}$. Equation (2.8) has the Painlevé property with singularities of the form

$$V = \phi^{-1} \sum_{j=0}^{\infty} V_j \phi^j \ , \tag{2.9}$$

and resonances at $j = -1, 2$. The Bäcklund transformation for Eq. (2.8)

$$V = (V_0/\phi) + V \tag{2.10}$$

obtains, with $b = 1$,

$$\begin{aligned} V_0^2 &= \phi_x\phi_t \ , \\ V_1 &= -\frac{1}{2}\frac{\phi_{xt}}{V_0} - \frac{1}{2}\frac{\phi_{xt}}{\phi_x\phi_t}V_0 \ , \end{aligned} \tag{2.11}$$

and the following overdetermined system of equations for ϕ:

$$\begin{aligned} &\phi_x \frac{\partial}{\partial x}\Omega_1 + \phi_t \frac{\partial}{\partial x}\Omega_2 = 4a(\phi_x\phi_t)^{1/2} \ , \\ &\Omega_1\Omega_2 = 0 \ , \end{aligned} \tag{2.12}$$

where Ω_1 and Ω_2 are defined by (2.6). From the identity

$$\phi_x \frac{\partial}{\partial x}\Omega_1 = \phi_t \frac{\partial}{\partial x}\Omega_2 \ , \tag{2.13}$$

Eqs. (2.12) have only the trivial solution and, as a consequence, the Bäcklund transformation (2.10) does not exist. This corresponds to the general result that Eq. (2.7) is known not to have a Bäcklund transformation [14], [16], although it does have a Lax pair [14, 15] and is completely integrable.

To proceed further we note the following direct formulation of Eq. (2.2) (sine-Gordon equation) in terms of the Schwarzian derivative. With

$$V = \phi_x \ , \tag{2.14}$$

we find that

$$\frac{\partial}{\partial t}\{\phi; x\} = -\frac{\partial}{\partial x}\left(\frac{1}{\phi_x}\right) \ . \tag{2.15}$$

From the symmetry $V \to 1/V$ of (2.2) we also have

$$V = 1/\psi_x = \phi_x \ , \tag{2.16}$$

$$\frac{\partial}{\partial t}\{\psi; x\} = -\frac{\partial}{\partial x}\left(\frac{1}{\psi_x}\right) \ . \tag{2.17}$$

The symmetry (2.16) is identical to (1.9) for the Schwarzian formulation (1.5) of the KdV sequence. With reference to the KdV sequence (1.1) Eq. (2.15) is identically

$$U_t + \frac{\partial}{\partial x} b^{-1}(U) = 0 \ , \tag{2.18}$$

where

$$U = \{\phi; x\} \ , \qquad b^{-1} = 1/\phi_x \ . \tag{2.19}$$

From Eqs. (1.11)–(1.13), the minus-one functional $b^{-1}(U)$, with $U = \{\phi; x\}$ satisfies the condition

$$(D - (\phi_{xx}/\phi_x))D(+(\phi_{xx}/\phi_x))b^{-1}(U) = 0 \ , \tag{2.20}$$

or

$$b^{-1} = \frac{a}{\phi_x} + b\frac{\phi}{\phi_x} + c\frac{\phi^2}{\phi_x} \ , \tag{2.21}$$

which obtains Eqs. (2.18) and (2.19) with

$$a = l \ , \qquad b = c = 0 \ .$$

In other words, the sine-Gordon equation is a specialization of the minus-one KdV equation. A comparison of the respective formulations (2.5), (2.6) and (2.14), (2.15), where the variable ϕ is *not* identified as the same in each, yields

$$\left(\frac{V_{2x}}{V_2} + \frac{V_x}{V}\right) = 2\sigma\left(\frac{V^{1/2}}{V_2^{1/2}} - \frac{V_2^{1/2}}{V^{1/2}}\right) \ , \tag{2.22}$$

where, in Eq. (2.6),

$$\beta = -2\sigma^2 \ .$$

Equation (2.22) is the classical BT for the sine-Gordon equation [5].

Now, for the Bullough-Dodd equation (2.7), we let

$$e^u = \phi_x \ , \tag{2.23}$$

and find the equation

$$\frac{\partial}{\partial t}\{\phi; x\} = -\frac{3}{2} b \frac{\partial}{\partial x}\left(\frac{1}{\phi_x^2}\right) \ . \tag{2.24}$$

The substitution

$$e^{-2u} = \psi_x \ , \tag{2.25}$$

gives us

$$\frac{\partial}{\partial t}\{\psi; x\} = -6a \frac{\partial}{\partial x}\psi_x^{-1/2} \ . \tag{2.26}$$

With reference to the CDG sequence (1.16), (1.17), we have, for Eqs. (2.24) and (2.26), respectively,

$$A_t + \theta_2 H_{-2}(A) = 0 \ , \tag{2.27}$$

$$U_t + \theta_1 G_{-2}(U) = 0 \ , (2.28)$$

where

$$A = \{\phi; x\} \ , \qquad U = \{\psi; x\} \ . \tag{2.29}$$

From (1.18)–(1.23) with

$$V = \phi_{xx}/\phi_x \ , \qquad W = \psi_{xx}/\psi_x \ , \tag{2.30}$$

$$\theta_2 H_{-1}(A) = \frac{\partial}{\partial x}\left\{\frac{1}{\phi_x^2}(a' + b'\phi + c'\phi^2 + d'\phi^3 + e'\phi^4)\right\} \ , \tag{2.31}$$

$$\theta_1 G_{-2}(U) = \frac{\partial}{\partial x}\{\psi_x^{-1/2}(a' + b'\psi)\} \ , \tag{2.32}$$

where (a', b', c', d', e') are numerical constants. Thus, Eqs. (2.24) and (2.26) are specializations of the minus (two) CDG equations. From equations (1.25) and (1.26), we conclude that

$$\begin{aligned} A_3 &= \{\phi; x\} \ , \\ A_2 &= -\frac{1}{2}\left(\frac{\psi_{xxx}}{\psi_x} - \frac{3}{4}\frac{\psi_{xxx}^2}{\psi_x^2}\right) \ , \end{aligned} \tag{2.33}$$

$$\begin{aligned} U_3 &= \{\psi; x\} \ , \\ U_2 &= -2(\phi_{xxx}/\phi_x) \end{aligned} \tag{2.34}$$

are solutions of Eqs. (2.27) and (2.28), respectively, and Eqs. (2.24) and (2.26) are connected by the transformation

$$\psi_x = \phi_x^{-2} \ . \tag{2.35}$$

However, without the invariance under the Möbius group

$$phi = (a\phi' + b)/(c\phi' + d) \tag{2.36}$$

for Eqs. (2.24) and (2.26), the Bäcklund transformation (1.24) does not exist for Eqs. (2.27) and (2.28).

From the results of [7] the CDG sequence is a consistent reduction of the Boussinesq sequence. There, the modified Boussinesq sequence is found to be

$$\begin{pmatrix} \theta \\ Z \end{pmatrix}_t = L^n \Omega_2 \begin{pmatrix} Z_x + \theta Z \\ -S - \frac{3}{2} Z^2 \end{pmatrix} \ , \tag{2.37}$$

where

$$\begin{aligned} S &= \theta_x - \tfrac{1}{2}\theta^2 \ , \\ \Omega_2 &= \begin{pmatrix} D & 0 \\ 0 & \frac{1}{3} D \end{pmatrix} \ , \end{aligned} \tag{2.38}$$

$$\begin{aligned} L &= \Omega_2 B^* \Omega_1^{-1} B \ , \\ \Omega_1^{-1} &= \begin{pmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{pmatrix} \ , \end{aligned} \tag{2.39}$$

$$B = \frac{1}{3} \begin{pmatrix} D - \theta & 3(D - Z) \\ -(D - 2Z)(D - \theta) & D^2 - 3DZ + 3Z^2 + 2(\theta_x - \frac{1}{2}\theta^2) \end{pmatrix} \ , \tag{2.40}$$

and B^* is the adjoint to B. Letting

$$\theta = \phi_{xx}/\phi_x \ , \qquad Z = \beta_{xx}/\beta_x \ , \tag{2.41}$$

we find that

$$\begin{pmatrix} \theta \\ Z \end{pmatrix}_t = L^{-2} \Omega_2 \begin{pmatrix} Z_x + \theta Z \\ -S - \frac{3}{2} Z^2 \end{pmatrix} = L^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.42}$$

is the equation

$$\frac{\partial}{\partial t} \begin{pmatrix} \phi_{xx}/\phi_x \\ \beta_{xx}/\beta_x \end{pmatrix} = \begin{pmatrix} a\phi_x^{-1/2}\beta_x^{-3/2} - b\phi_x^{-1/2}\beta_x^{3/2} + c\phi_x \\ a\phi_x^{-1/2}\beta_x^{-3/2} + b\phi_x^{-1/2}\beta_x^{3/2} \end{pmatrix} \ . \tag{2.43}$$

Let

$$c = 2\ , \qquad b = 1\ , \qquad a = -1\ , \tag{2.44}$$

$$\begin{aligned} \phi_x &= \phi_{1x}/\phi_{3x}\ , \qquad \beta_x = 1/\phi_{2x}\ , \\ W_i &= \phi_{ixx}/\phi_{ix}\ , \qquad \phi_{1x}\phi_{2x}\phi_{3x} = 1\ , \end{aligned} \tag{2.45}$$

and $\sum_1^3 W_i = 0$. This gives us, from Eq. (2.43), the three component Toda equation

$$\frac{\partial}{\partial t}\widehat{W} = \begin{pmatrix} \phi_{1x}/\phi_{3x} - \phi_{2x}/\phi_{1x} \\ \phi_{2x}/\phi_{1x} - \phi_{3x}/\phi_{2x} \\ \phi_{3x}/\phi_{2x} - \phi_{1x}/\phi_{3x} \end{pmatrix}\ . \tag{2.46}$$

With

$$\phi_{ix} = e^{\theta i}\ , \tag{2.47}$$

this is

$$\theta_{ixt} = e^{\theta i - \theta i-1} - e^{\theta i+1-\theta i}\ , \tag{2.48}$$

for $i = 1, 2, 3, \ldots$ (mod N), where

$$\sum^{N} \theta_{ix} = 0\ , \qquad N = 3\ .$$

Thus, the three component Toda lattice is the minus-one equation of the Boussinesq sequence. A Bäcklund transformation is known to exist for Eq. (2.48) [14]. However, whether a Bäcklund transformation can be constructed for one of the equivalent forms of Eq. (2.48) by the Painlevé method is nearly a moot point. For instance, in Eq. (2.43), let

$$\begin{aligned} U &= \ln(\beta_x^{3/2}\phi_x^{1/2})\ , \\ \theta &= \ln(\beta_x^{3/2}\phi_x^{-1/2})\ , \end{aligned} \tag{2.49}$$

and find the equation

$$U_{xt} = 2ae^{-u} + be^{\theta} + (c/2)e^{u-\theta}\ , \tag{2.50}$$

$$\theta_{xt} = 2ae^{-u} + 2be^{\theta} - (c/2)e^{u-\theta}\ . \tag{2.51}$$

With

$$W = e^{u}\ , \qquad V = e^{-\theta}\ ,$$

we find the system

$$\begin{aligned} V(WW_{xt} - W_xW_t) &= 2aVW + bW^2 + (c/2)V^2W^3\ , \\ W(VV_{xt} - V_xV_t) &= -aV^2 - 2bVW + (c/2)V^3W^2\ . \end{aligned} \tag{2.52}$$

Equations (2.52) have singularities of the form

$$
\begin{aligned}
W &= \phi^{-1} \sum W_j \phi^j \ , \\
V &= \phi^{-1} \sum V_j \phi^j \ ,
\end{aligned}
\tag{2.53}
$$

with resonances at

$$
j = -1, 0, 1, 2 \ . \tag{2.54}
$$

The Bäcklund transformation

$$
\begin{aligned}
W &= W_0 \phi^{-1} + W_1 \ , \\
V &= V_0 \phi^{-1} + V_1 \ ,
\end{aligned}
\tag{2.55}
$$

taking into account (2.54), produces a system of nine equations for five functions, $(\phi, W_0, V_0, W_1, V_1)$. An analysis of this system indicates that the BT (2.55) determines a reduction of Eq. (2.52):

$$
V = \lambda^2 W \ , \tag{2.56}
$$

where

$$
\lambda^2 = -b/a \ . \tag{2.57}
$$

The reduced system is Eq. (2.8), the Bullough-Dodd equation, for which a Bäcklund transformation of the form (2.10), (2.55) does not exist. This result is similar to that of [7], where the BT for the modified nonlinear Shrödinger (NLS) equations determined a reduction to Burgers equation. The BT's defined by other forms of the singularities for equations (2.52) and equivalent systems, have not been investigated, these typically being highly overdetermined and implicit systems of equations.

The situation here contrasts sharply with the analysis for the (positive) Boussinesq sequence. For the (positive) Boussinesq sequence the system of equations produced by the (Painlevé) BT is *not* overdetermined. This is the result of the distribution of the resonances and the linearity of the highest derivative for the positive sequence. Therefore, from the point of view of Painlevé analysis and the calculation of Bäcklund transformations, it is of interest to identify when a system is a defined by a negative functional of a sequence of equations. The results of [17] demonstrate that all the (N-component) two-dimensional, periodic Toda lattice equations can be identified with the minus-one functionals of equation sequences. Therefore, by suitably developing the recursion operators for these sequences, it should be possible to recursively define the (Painlevé) BT's. On the other hand, a different approach to Bäcklund transformations and the Painlevé property for the Toda lattice is presented in [18].

In terms of the variables W_i defined by Eqs. (2.45) for the (modified) Toda equation (2.46), the recursion operator for the Boussinesq sequence assumes a considerably more symmetric form. That is, with

$$\widehat{W} = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} , \tag{2.58}$$

the modified Boussinesq sequence is

$$\widehat{W}_t = L^n \widehat{W}_x , \tag{2.59}$$

where

$$\begin{aligned} L &= \Omega J \\ \Omega &= \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} , \\ D &= \frac{\partial}{\partial x} , \end{aligned} \tag{2.60}$$

and for $i, j = 1, 2, 3$,

$$J = \{J_{ij}\} , \tag{2.61}$$

$$J_{ii} = -16 W_i D - 8 W_{ix} - 8 W_i D^{-1} A_i - 8 A_i D^{-1} W_i , \tag{2.62}$$

$$A_i = (W_{i+1} - W_{i-1})_x - (W_{i+1} - W_{i-1})^2 - 6 W_{i+1} W_{i-1} , \tag{2.63}$$

where $i = 1, 2, 3, \ldots$ (mod 3). And

$$J_{12} = 8D^2 - 16 W_3 D - 8 B_3 - 8 W_1 D^{-1} A_2 - 8 A_1 D^{-1} W_2 , \tag{2.64}$$

$$J_{13} = -8D^2 - 16 W_2 D - 8 C_2 - 8 W_1 D^{-1} A_3 - 8 A_1 D^{-1} W_3 , \tag{2.65}$$

$$J_{23} = -8D^2 - 16 W_1 D - 8 B_1 - 8 W_2 D^{-1} A_3 - 8 A_2 D^{-1} W_3 , \tag{2.66}$$

$$J_{21} = -J_{21}^* , \qquad J_{31} = -J_{13}^* , \qquad J_{32} = -J_{23}^* , \tag{2.67}$$

$$\begin{aligned} B_3 &= W_{3x} + (W_1 - W_3)^2 + 3 W_1 W_3 , \\ B_1 &= W_{1x} + (W_1 - W_3)^2 + 3 W_1 W_3 , \\ C_2 &= W_{2x} - W_2^2 + W_1 W_3 . \end{aligned} \tag{2.68}$$

It is to be remarked that

$$J^* = -J , \tag{2.69}$$

and, by (2.45),

$$\sum_1^3 W_i = 0 \tag{2.70}$$

implies that

$$\sum_{i=1}^3 J_{ij} = 0 \ , \quad \text{for } j = 1,2,3 \ , \tag{2.71}$$

$$\sum_{j=1}^3 J_{ij} = 0 \ , \quad \text{for } i = 1,2,3 \ . \tag{2.72}$$

From Eqs. (2.46) and (2.60), the modified Toda lattice equations are

$$L \circ \widehat{W}_t = 0 \ , \tag{2.73}$$

or the recursion operator annihilates the right side of the modified Toda lattice equation (2.46). This suggests that explicit formulas for recursion operators of the Toda Lattice equations can be constructed from a suitable system of annihilators expressed in terms of the variables $\{W_i\}$, (2.45). It is not difficult to find formulas for the annihilators [for any N in Eq. (2.48)]. Yet it is nontrivial to verify that the resulting expression is the recursion operator for a sequence. After verification of this procedure it then is necessary to find a transformation analogous to (2.45) for the Boussinesq three-component sequence in which (1) [unlike the Toda formulation (2.46)] all variables allow simultaneous singularities, and (2) a component of the transformed system is invariantly formulated (in terms of the Schwarzian derivative) thereby allowing an analysis similar to that for the Boussinesq sequence [7]. It would be most interesting to resolve the question of Schwarzian formulation through construction of Bäcklund transformations. It is our view that the Schwarzian derivative arises naturally from the essential dependence of the singularities on one preferred (spacelike) independent variable (x), and not, say, from the order of the monodromy group of the associated Lax operator. In this connection, the Schwarzian derivative expresses the differential invariance of an equation when subject to the natural (unique) group of conformal transformations preserving the complex sphere (C^1). When the structure of an equation's singularities depends essentially on more than one complex independent variable, various generalizations of the Schwarzian derivative are indicated.

Finally, the minus-one equation of the Hirota-Satsuma [5], [19] sequence can be shown to be equivalent to the system

$$\begin{aligned} A_{xt} &= ae^{A-B} + be^{-A-B} \ , \\ B_{xt} &= -ae^{A-B} + be^{-A-B} - e^{B} \end{aligned} \tag{2.74}$$

of Toda type. Equation (2.74) is reduction of the four-component Toda lattice presented in [14]. With reference to Eq. (2.43), the minus-one Boussinesq equation tan be written as

$$\begin{aligned} C_{xt} &= ae^{C-D} + be^{-C-D} \ , \\ D_{xt} &= -ae^{C-D} + be^{-C-D} + cd^{2D} \ , \end{aligned} \tag{2.75}$$

where

$$C = \ln(\beta_x^{3/2}) \ , \qquad D = \ln(\phi_x^{1/2}) \ . \tag{2.76}$$

3 Uniformization of the Harry Dym sequence

With reference to Eqs. (1.1)–(1.9), the KdV equation

$$U_t + \frac{\partial}{\partial x}(U_{xx} + \tfrac{3}{2}U^2) = 0 \tag{3.1}$$

has the Bäcklund transformation

$$U = 4\frac{\partial^2}{\partial x^2}\ln\phi + U_2 \ , \tag{3.2}$$

where

$$U_2 = -\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \ , \tag{3.3}$$

and

$$\phi_t/\phi_x + \{\phi; x\} = \lambda \ . \tag{3.4}$$

The Schwarzian derivative

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\tfrac{\phi_{xx}}{\phi_x} \right) - \frac{1}{2}\left(\tfrac{\phi_{xx}}{\phi_x} \right)^2 \tag{3.5}$$

is invariant under the Möbius group

$$\phi = (a\psi + b)/(c\psi + d) \ , \tag{3.6}$$

and it tan be shown [4] that Eq. (3.4) is invariant under (3.6) and the transformation

$$\phi_x = \beta_x^{-1} \ . \tag{3.7}$$

Furthermore,

$$\begin{aligned} \{\phi;x\} &= \{\psi;x\} + \{f;\psi\}\psi_x^2 \ , \\ \{\phi;x\} &= h'^2\{\psi;z\} + \{h;x\} \ , \end{aligned} \tag{3.8}$$

where

(i) $\phi = f(\psi)$

is an arbitrary change of dependent variable, and

(ii) $z = h(x)$

is an arbitrary change in independent variable. By the transformation properties of the Schwarzian derivative,

$$\{\phi;x\} = -\phi_x^2\{x;\phi\} \ . \tag{3.9}$$

Under the change of variables

$$x \to \phi \ , \qquad t \to t \ , \qquad \phi \to x \ , \tag{3.10}$$

$$\phi_x = 1/x_\theta \ , \tag{3.11}$$

and

$$x_t = -\phi_t/\phi_x \ .$$

Equation (3.4) becomes [2]

$$x_\phi^2 x_t + \lambda x_\phi^2 + \{x;\phi\} = 0 \ , \tag{3.12}$$

which is invariant under the change of independent variable (3.6). Equation (3.12) is equivalent to the Harry Dym equation [2]. Equation (3.12) is also invariant under the transformation (implicit change of independent variable) (3.7) or

$$x_\phi x_\beta = 1 \ . \tag{3.13}$$

From (3.13)

$$\begin{aligned} \phi_\beta &= x_\beta/x_\phi = x_\beta^2 \ , \\ \beta_\phi &= x_\phi^2 \ , \end{aligned} \tag{3.14}$$

and using (3.8) or directly from (3.12)

$$\beta_t + \beta_\phi^{-1/2}\{\beta;\phi\} = 0 \ . \tag{3.15}$$

By the involution (3.13) or directly,

$$\phi_t + \phi_\beta^{-1/2}\{\phi;\beta\} = 0 \ . \tag{3.16}$$

Therefore, under the inversion

$$(\phi, \beta) \to (\beta, \phi) \ , \tag{3.17}$$

Eqs. (3.15) and (3.16) are invariant. However, unlike Eq. (3.12), Eq. (3.15) is not invariant under the Möbius group (3.6). The equivalent Bäcklund transformation for (3.15) is defined by

$$h_\phi = \phi^2 \beta_\phi \ , \qquad \phi = -1/\psi \ , \tag{3.17$'$}$$

where it can be shown that $h = h(\psi)$ satisfies

$$h_t + h_\psi^{-1/2} \{h; \psi\} = 0 \ . \tag{3.17$''$}$$

By composition of the BT (3.17) and (3.17′) various sequences of solutions can be constructed. Equation (3.15) may also be written as

$$\beta_t = 2 \frac{\partial^2}{\partial \phi^2} (\beta_\phi^{-1/2}) \ , \tag{3.18}$$

and

$$W = \beta_\phi = x_\phi^2 \tag{3.19}$$

obtains the Harry Dym equation

$$W_t = 2 \frac{\partial^3}{\partial \phi^3} W^{-1/2} \ . \tag{3.20}$$

Equations (3.12), (3.15), and (3.20) all allow movable branch point singularities (even logarithmic singularities)and thus do not directly possess the Painlevé property. For instance, Eq. (3.12) has singularities of the form

$$x = x_0 \epsilon^{1/3} + x_1 \epsilon + x_2 \epsilon^{5/3} + \cdots \ , \tag{3.21}$$

where

$$\epsilon = \epsilon(\phi, t)$$

represents the singularity manifold. For this reason, it is not possible to directly calculate BT's for, say, Eq. (3.15) by an expansion about the singular manifold. However, solutions of Eqs. (3.15) and (3.16) can be represented implicitly through solutions of Eq. (3.4) that satisfy the condition (3.7). From the rational solutions of Eq. (3.4) [4], the expressions

(i) $\qquad \phi = 1/x \ , \qquad \beta = x^3/3 + 4t \ ,$ (3.22)

$$\text{(ii)} \qquad \phi = \frac{1}{x^3/3 + 4t} \ , \qquad \beta = \frac{x^6 + 60tx^3 - 720t^2}{5x} \tag{3.23}$$

implicitly define solutions of (3.15) and (3.16). In the above, (ϕ, β) are meromorphic functions of a "uniformizing" variable x (and t). In general, Eq. (3.15) [or (3.16)] has a uniform representation

$$\phi = f(x,t) \ , \qquad \beta = g(x,t) \ , \tag{3.24}$$

where f and g satisfy Eq. (3.4) and

$$f_x g_x = 1 \ . \tag{3.25}$$

Since Eqs. (3.15) and (3.16) are invariant under inversion (3.17) of independent/dependent variables, the uniformizing functions (f, g) satisfy the same differential equation. For an arbitrary differential equation it would be necessary to make a simultaneous substitution of dependent and independent variables subject to the requirement that the "uniformizing" substitutions be "meromorphic" functions of the uniform variable(s). The resulting set of differential equations (possessing the Painlevé property) may then be studied directly by use of the Bäcklund transformations.

The KdV sequence [Eq. (1.5)] is

$$\phi_t/\phi_x + b^n(\{\phi; x\}) = 0 \ , \tag{3.26}$$

where

$$\begin{aligned} &D_x b^{n+1} = (D_x^3 + 2UD_x + U_x) b^n \ , \\ &b^n = b^n(U) \ , \end{aligned} \tag{3.27}$$

$$\begin{aligned} &D_x = \frac{\partial}{\partial x} \ , \\ &b^0 = 1 \ , \qquad b^1 = U \ , \qquad b^2 = U_{xx} + \tfrac{3}{2}U^2 \ . \end{aligned} \tag{3.27$'$}$$

With

$$\begin{aligned} &V = \phi_{xx}/\phi_x \ , \\ &U = V_x - \tfrac{1}{2}V^2 = \{\phi; x\} \ , \end{aligned} \tag{3.28}$$

the Lenard formula (3.27) is

$$D_x b^{n+1} = (D_x - V) D_x (D_x + V) b^n \tag{3.29}$$

or

$$b^{n+1} = L_v \circ b^n \ , \tag{3.30}$$

where

$$L_v = D_x^{-1}(D_x - V)D_x(D_x + V) \ .$$

From the identification (3.28),

$$L_v = D_x^{-1}\phi_x D_x \phi_x^{-1} D_x \phi_x^{-1} D_x \phi_x \ , \tag{3.31}$$

and, by (3.26)

$$\phi_t/\phi_x + L_v^n \circ 1 = 0 \ , \tag{3.32}$$

where $L_v^n = L_n \circ L_v^{n-1}$ and (3.32) are invariant under (3.6) and (3.7). Equations (3.32) can be written as "Hamiltonian" systems

$$\phi_t + M_v^n \circ \phi_x = 0 \ , \tag{3.33}$$

where

$$M_v = \Omega_1 J_1 \ , \tag{3.34}$$

$$\begin{aligned} \Omega_1 &= \phi_x D_x^{-1} \phi_x \ , \\ J_1 &= D_x \phi_x^{-1} D_x \phi_x^{-1} D_x \ . \end{aligned} \tag{3.35}$$

Under the change of variable (3.10) and (3.11), and using

$$D_\phi = x_\phi D_x = \phi_x^{-1} D_x \ , \tag{3.36}$$

we find

$$x_t = L_\phi^n \circ 1 \ , \tag{3.37}$$

where

$$L_\phi = \Omega_2 J_2 \ , \tag{3.38}$$

$$\Omega_2 = D_\phi^{-1} \ , \qquad J_2 = x_\phi^{-1} D_\phi^3 x_\phi^{-1} \ . \tag{3.39}$$

Equations (3.37) are Hamiltonian and invariant under (3.6) and (3.13). From (3.13) and (3.14),

$$x_\phi = \beta_\phi^{1/2} \ , \tag{3.40}$$

and

$$\beta_t = \Phi_\phi^n \circ 1 \ , \tag{3.41}$$

$$\Phi_\phi = \Omega_3 J_3 \ , \tag{3.42}$$

$$\begin{aligned} \Omega_3 &= D_\phi \ , \\ J_3 &= D_\phi \beta_\phi^{-1/2} D_\phi^{-1} \beta_\phi^{-1/2} D_\phi \ . \end{aligned} \tag{3.43}$$

Under the inversion (3.17), the identity

$$\beta_\phi \Phi_\beta = \Phi_\phi \beta_\phi \tag{3.44}$$

implies

$$\phi_t = \Phi_\beta^n \circ 1 \ , \tag{3.45}$$

or the sequence (3.41) is invariant under inversion (3.17) [and also under (3.17′)]. The Harry Dym sequence

$$W = \beta_\phi \ , \qquad W_t = H_\phi^n \circ 1 \ , \tag{3.46}$$

where

$$H_\phi = \Omega_4 J_4 \ , \tag{3.47}$$

$$\Omega_4 = D_\phi^3 \ , \qquad J_4 = W^{-1/2} D_\phi^{-1} W^{-1/2} \ , \tag{3.48}$$

is Hamiltonian and, by the above identification, is uniformly represented by the KdV sequence (3.33).

A similar procedure for the Caudrey-Dodd-Gibbon sequence (1.27) obtains the sequence

$$x_t = M_\phi^n \circ 1 \ , \tag{3.49}$$

where

$$\begin{aligned} M_\phi &= J_2 \theta_2 \ , \\ J_2 &= D_\phi^{-1} x_\phi^{-2} D_\phi^5 x_\phi^{-2} D_\phi^{-1} \ , \\ \theta_2 &= x_\phi^{-1} D_\phi^3 x_\phi^{-1} \ , \end{aligned} \tag{3.50}$$

and J_2, θ_2 are identified by Eqs. (1.23). We note the relative simplicity of the recursion operators (3.38) and (3.50) for the "automorphic" sequences (3.37) and (3.49) [the term "automorphic" indicating the invariance of these equations under the Möbius group (3.6)].

The classical theory of uniformization by automorphic functions is concerned with algebraic functions (Riemann surfaces) defined by polynomial in two (complex) variables [20], [21]

$$P(z, w) = 0 \ , \tag{3.51}$$

where P is irreducible (not the product of two polynomials of lower degree). Equation (3.51) defines (in general) w as the multiple-valued function of z (or z as a multiple-valued function of w); for instance, when

$$z^2 + w^2 = 1 \ , \qquad w = \pm(1 - z^2)^{1/2} \ . \tag{3.52}$$

Yet, it is also possible to represent the solutions of (3.52)as

$$\text{(i)} \qquad z = \sin t \ , \qquad w = \cos t \ , \tag{3.53}$$

or

$$\text{(ii)} \qquad z = 2t/(1+t^2) \ , \qquad w = (1-t^2)/(1+t^2) \ , \tag{3.54}$$

where in both cases (z, w) are single-valued functions t [in (3.54) rational functions of t]. Equations (3.53) and (3.54) define uniformizations of Eq. (3.52), i.e., uniform, single-valued.

A general result [20] is an algebraic function (3.51) can be uniformized by means of (i) rational functions if the genus of (3.51) is zero ($p = 0$), (ii) elliptic functions if the genus is one ($p = 1$), and (iii) Fuchsian functions of the first kind when the genus is greater than one ($p > 1$).

A Fuchsian function is a meromorphic function that is (automorphic) invariant under a "discrete" subgroup of the Möbius group. The discrete (properly discontinuous) sub-group is required to preserve (carry into itself) a circle (the "principal" circle) and map interior points (of the circle) into interior points and exterior points into exterior points. That is, if M is the discrete subgroup

$$M \subset M_{0b}$$

and the principal circle is $|t| = 1$,

$$t' = (at + b)/(ct + d) \in M \ ,$$

then $f(t') = f(t)$, where

$$|t| < 1 \Leftrightarrow |t'| < 1 \ ,$$
$$|t| = 1 \Leftrightarrow |t'| = 1 \ ,$$

and $f(t)$ is meromorphic in $|t| < 1$.

Finally, a Fuchsian function of the first kind is a Fuchsian function for which the limit points of the associated group are dense on the principal circle, and hence, the principal circle is a natural boundary for the function. For a discussion of the above, the reader is advised to consult [20] and [21]. In general, the theory of uniformization, especially as formulated by Teichmuller, Alfors, and Bers [22]–[24], determines the existence of various forms of uniformizations for general classes of Riemann surfaces. Yet, actual uniformizations for Eqs. (3.51) are known in only a few Special cases [25]. That is, representations of the group and the associated automorphic functions have been calculated for only a finite number of equations of the form (3.51). Of special interest in this regard is Whittaker's conjecture, our account of which is taken from [25].

Whitaker considered the hyperelliptic equations

$$w^2 = (z - e_1)(z - e_2) \cdots (z - e_{2p+2}) = f(z) \ , \tag{3.55}$$

where the $2p + 2\{e_j\}$ are distinct complex numbers. It is first shown that the group Γ of Eq. (3.55) is a subgroup of index 2 in a larger group Γ^*, in which the fundamental region for Eq. (3.55) has genus zero. Therefore, by a general result, any automorphic function attached to Γ^* is a rational function of one such automorphic function $z(t)$. The Schwarzian derivative of an automorphic function is itself an automorphic function and therefore must be a rational function of z. That is,

$$\{t; z\} = -z_t^{-2}\{z; t\} = R(z) \tag{3.56}$$

is automorphic (in t) and it can be shown that

$$R(z) = \frac{3}{8}\left[\sum_{n=1}^{2p+2} \frac{1}{(z - e_n)^2} - \frac{g(z)}{f(z)}\right] \ , \tag{3.57}$$

where $f(z)$ is defined by (3.55) and $g(z)$ is of the form

$$g(z) = 2(p+1)z^{2p} - 2p \sum_{n=1}^{2p+2} e_n z^{2p-1} + \sum_{k=0}^{2p-2} C_k z^{2p-2-k} \ . \tag{3.58}$$

The $2p-1$ coefficients $\{C_k\}$ are the accessory parameters and are determined by the group Γ^*.

If $R(z)$ is determined, then it is known that by the substitution

$$t = (\eta_1/\eta_2)(z) \ , \tag{3.59}$$

Eq. (3.56) becomes the linear differential equation

$$\frac{d^2\eta}{dz^2} + \frac{1}{2}R(z)\eta = 0 \ , \tag{3.60}$$

and Γ^* is the monodromy group of (3.60). Therefore to find Γ^* (and Γ) and $z(t)$ (the uniformization), it is necessary to determine $R(z)$ (the accessory parameters) $\{C_k\}$. Motivated by certain scaling arguments, Whittaker conjectured that

$$R_z = \frac{3}{8}\left[\left(\frac{f'(z)}{f(z)}\right)^2 - \frac{(2p+2)f''(z)}{(2p+1)f(z)}\right] \ . \tag{3.61}$$

It can be shown that if the $\{C_k\}$ are polynomial functions of $\{e_n\}$ then (3.61) is correct for the general hyperelliptic equation (3.55).

As is indicated by the above, the Schwarzian derivative, through its connection with the automorphic functions, arises in many problems related to the study of Riemann surfaces [26], [27]. For instance, the space of moduli (accessory parameters) of Riemann surfaces, the so called universal Teichmuller space, is equivalent to a Banach space of bounded Schwarzian derivatives of univalent functions [23]. In a different direction, Burchnall and Chaundy [28] have shown that a commuting pair of linear differential operators

$$PQ = QP \ , \tag{3.62}$$

where $\deg(P) = m$, $\deg(Q) = n$, satisfy an algebraic identity

$$f(P, Q) = 0 \ , \tag{3.63}$$

with constant coefficients, where f is of degree n in P and of degree m in Q. They remark that (3.62) and (3.63) provide a form of uniformization for the algebraic curve $f(z, w)$. Since the work of Burchnall and Chaundy is the forerunner of later developments in the theory of integrable systems (Lax pairs), there may be connections between the classical (and recent) theory of uniformization (Riemann surfaces, automorphic functions) and the theory of integrable systems (isomonodromy deformations, finite zone potentials).

Now, with regard to the uniformization of the Harry Dym sequences (3.37), (3.41), and (3.46) by the (Schwarzian) KdV sequence (3.32), we remark that, since ϕ is meromorphic as a function of x and x is (loosely speaking) "automorphic" with regard to ϕ [Eqs. (3.37) being invariant under the Möbius group], the formulation is not standard in terms of the theory of uniformization (where the meromorphic and automorphic dependence refer to the same variable). We can, of course, claim that ϕ is automorphic with regard to the identity group acting on the independent variable x but this somehow seems to miss the point. Instead, it seems worthwhile to enquire whether Eqs. (3.37) allow solutions that are automorphic (invariant under a discrete subgroup of the Möbius group acting on ϕ) without requiring meromorphicity.

Toward this goal, consider Eq. (3.12), where without loss of generality, $\lambda = 0$,

$$x_t + x_\phi^{-2}\{x; \phi\} = 0 \ , \tag{3.64}$$

which is equivalent to

$$x_t = D_\phi^{-1} x_\phi^{-1} \frac{\partial^3}{\partial \phi^3} x_\phi^{-1} \ , \tag{3.65}$$

where

$$D_\phi = \frac{\partial}{\partial \phi} \ .$$

The first two conserved quantities of Eq. (3.64) are

$$\begin{aligned} C_1 &= \oint x_\phi^2 d\phi \ , \\ C_2 &= \oint \frac{x_{\phi\phi}^2}{x_\phi^3} d\phi \ . \end{aligned} \tag{3.66}$$

From the recursion operator for the sequence (3.37),

$$\begin{aligned} L_\phi &= \Omega_2 J_2 \ , \\ \Omega_2 &= D_\phi^{-1} \ , \\ J_2 &= x_\phi^{-1} D_\phi^3 x_\phi^{-1} \ , \end{aligned} \tag{3.67}$$

the functional gradients of the conserved quantities b^j satisfy the equation

$$b^{j+1}(x_\phi) = M_\phi b^j(x_\phi) \ , \tag{3.68}$$

where

$$M_\phi = J_2 \Omega_2 \ , \qquad b^0 = 0 \ . \tag{3.69}$$

The Lax pair for Eqs. (3.64) and (3.65) is

$$\begin{aligned} Y_{\phi\phi} &= \lambda x_\phi^2 Y \ , \\ Y_t &= -4\lambda x \phi^{-1} Y_\phi + 2\lambda (x_\phi^{-1})_\phi Y \ . \end{aligned} \tag{3.70}$$

Equations (3.70) have a curious property in that if $Y(\phi)$ is a solution, then

$$W(\epsilon) = \epsilon Y(1/\epsilon) \tag{3.71}$$

is a solution of

$$\begin{aligned} W_{\epsilon\epsilon} &= \lambda x_\epsilon^2 W \ , \\ W_t &= -4\lambda x_\epsilon^{-1} W_\epsilon + 2\lambda (x_\epsilon^{-1})_\epsilon W \ , \end{aligned} \tag{3.72}$$

where

$$\epsilon = 1/\phi \ . \tag{3.73}$$

Therefore, from (3.71), (3.73), and the invariance of Eqs. (3.70) under scaling and translation,

$$\phi \to a\phi \ , \qquad \phi \to \phi + b \ , \tag{3.74}$$

it is possible to observe the effect of the Möbius group on the eigenfunctions of (3.70).

Next we recall that equations (3.64) and (3.65) have movable singularities of the form

$$\text{(i)} \qquad x = \psi^{1/3} \sum_{j=0}^{\infty} x_j \psi^{j/3} \ , \tag{3.75}$$

with resonances at

$$j = -3, 2, 4$$

and

$$\psi = \psi(\phi, t) \ , \qquad \psi_\phi \neq 0 \ .$$

There are also singularities of the form

$$\text{(ii)} \qquad x = x_0\phi^{-1} + x_1 + x_2\phi + x_3\phi^2 + x_4\phi^3 + x_5 t\phi^4 + \cdots , \tag{3.76}$$

and when $m > 1$

$$\text{(iii)} \qquad x = x_0\phi^{-m} + \cdots + x_{3m} t\phi^{2m} + \cdots , \tag{3.77}$$

where the $\{x_j\}$ are constants. Furthermore, it tan be shown that the singularities (3.77) have an expansion of the form

$$x = \phi^{-m} \sum_{k=0}^{\infty} P_k(\phi) t^k \phi^{3km} \ , \tag{3.78}$$

where

$$P_k(\phi) = P_{0k} + P_{1k}\phi + \cdots + P_{3m-1,k}\phi^{3m-1} \ . \tag{3.79}$$

The locations of the singularities (3.76) and (3.77) do not depend on t but are movable, since the locations are not fixed by the equation. By the invariance of Eq. (3.64) under translation in ϕ, the singularities may be located at any point $\phi = \phi_0$, where ϕ_0 is constant. Also the singularities (3.76) and (3.77) refer to the behavior of ϕ as $x \to \infty$.

To obtain an automorphic solution it is consistent to set

$$x(\phi, 0) = x_0(\phi) \ ,$$

where, say,

$$z_0 = -\phi^{-1} + \phi \ . \tag{3.80}$$

By the invariance of Eqs. (3.64) and (3.80),

$$x(\phi, t) = x(\epsilon, t) \ , \tag{3.81}$$

where

$$\phi = -1/\epsilon \ . \tag{3.82}$$

With (3.80), the scattering operator in the Lax pair (3.70) has an irregular singular point at $\phi = 0$ and at $\phi = \infty$. Therefore, it is not (entirely) obvious how to implement (3.70) to find the solution (3.81). However, we claim that, with automorphic x, it is natural to study solutions of (3.70) on the boundary of a "fundamental domain" of the automorphism group. In this case, the boundary of the fundamental domain is, by (3.82),

$$|\phi| = 1 \ . \tag{3.83}$$

Using (3.80) and extending the path of integration along the contour (3.83), it is found, by the residue theorem applied to (3.66), that

$$C_1 = C_2 \equiv 0 \ . \tag{3.84}$$

In the same way all the conserved qualities of Eq. (3.64) can be evaluated.

On the other hand, the KdV data, defined by (3.80), is

$$U_0 = \{\phi; x\} = -6/(x^2+4)^2 \ , \tag{3.85}$$

which is within the class of initial data for which the inverse scattering on the real line can be solved. That is

$$\phi = V_1/V_2 \ , \tag{3.86}$$

where (V_1, V_2) satisfy

$$V_{xx} = (\lambda - U_0/2)V \ . \tag{3.87}$$

We note that the irregular singularities of (3.70) refer to the behavior of ϕ (and the KdV data) as $x \to \infty$. We shall defer further consideration of this problem at this time, except to note that (3.87) with U_0 defined by (3.85) (1) does not support solitons, and (2) can be solved in terms of hyper-geometric functions when $\lambda = 0$.

As an example of a system with finite-degree branch points, the equation

$$U_t + U^3 U_x + U_{xxx} = 0 \tag{3.88}$$

has singularities of the form

$$U = \psi^{-2/3} \sum_{j=0}^{\infty} U_j \psi^{j/3} \ , \tag{3.89}$$

with resonances at $j = -3, 8, 10$ and is known to be nonintegrable [29] (has only three conserved quantifies). Since the traveling wave solution

$$U + U(x + Ct) \tag{3.90}$$

can be found (as a hyperelliptic function) the presence of a nontrivial time dependence probably introduces additional singularities along the systems characteristics. On the other hand, in [30] (and the references cited therein) many examples of integrable systems with branch point behavior are found.

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Appendix A: the Caudrey-Dodd-Gibbon equation reconsidered

In [4] we have studied the Bäcklund transformation

$$U = \frac{\partial^2}{\partial x^2} \ln \phi + U_2 \tag{$A1$}$$

for the Caudrey-Dodd-Gibbon equation

$$U_t + \frac{\partial}{\partial x}(U_{xxxx} + 30UU_{xx} + 60U^3) = 0 \ . \tag{$A2$}$$

The resulting overdetermined system of equations for (ϕ, U_2) is

(i) $$\frac{\phi_t}{\phi_x} + \frac{\partial^2}{\partial x^2}\{\phi; x\} + 4\{\phi; x\}^2 + 5(\theta_{xx} + \theta^2 + 2\{\phi; x\}\theta) = 0 \ , \tag{A3}$$

(ii) $$\theta\theta_{xx} - \theta_x^2/2 + \tfrac{2}{3}\theta^3 + \{\phi; x\}\theta^2 = \lambda \ , \tag{A4}$$

(iii) $$6U_2 + \tfrac{3}{2}(\phi_{xx}^2/\phi_x^2) = \theta - \{\phi; x\} \ , \tag{A5}$$

or

$$6U_2 + \phi_{xxx}/\phi_x = \theta \ . \tag{$A6$}$$

In [4], due to a certain complexity in the analysis, we set

$$\theta = \lambda = 0 \tag{A7}$$

and found the Lax pair for Eq. (A2) in the instance when the spectral parameter vanishes. Later, the complete Lax pair was derived through a certain resummation of terms defined by (A1). In effect, the Lax pair was not derived directly from the BT (A1). In this appendix we will derive the Lax pair directly from the BT [i.e., Eqs. (A3)–(A6)], thereby showing that a resummation of terms is unnecessary. We present this result for the sake of completeness since, as shown earlier, the minus-one equation of the Caudrey-Dodd-Gibbon sequence does not have a Bäcklund transformation. The point here is that, in spite of this result, everything goes through without restriction for the CDG equation.

The first step is to "solve" Eq. (A4). Therefore, let

$$\lambda = -2\sigma^2 \ , \tag{A8}$$

$$\theta = \sigma(\beta/\beta_x) \ , \tag{A9}$$

and find that Eq. (A4) is

$$\{\phi; x\} + \tfrac{2}{3}\sigma(\beta/\beta_x) = \{\beta; x\} \ . \tag{A10}$$

From the formula for the Schwarzian derivative, if

$$\phi = V_1/V_2 \ , \tag{A11}$$

$$\Phi_x = V_2 V_{1x} - V_1 V_{2x} \ , \tag{A12}$$

then

$$\{\phi; x\} = \{\Phi; x\} + 2(V_{2x}V_{1xx} - V_{1x}V_{2xx})/\Phi_x \ . \tag{A13}$$

Comparing the identity (A13) with (A10) suggests that

$$\beta_x = \Phi_x = V_2 V_{1x} - V_1 V_{2x} \ , \tag{A14}$$

and

$$(\sigma/3)\beta = V_{1x}V_{2xx} - V_{2x}V_{1xx} \ . \tag{A15}$$

From (A14) and (A15)

$$V_{1x}V_{2xxx} - V_{2x}V_{1xxx} = (\sigma/3)(V_2 V_{1x} - V_1 V_{2x}) \ , \tag{A16}$$

which we solve by requiring that (V_1, V_2) satisfy

$$V_{xxx} = aV_x + (\sigma/3)V \ . \tag{A17}$$

From (A14)–(A17)

$$\beta_{xxx} = a\beta_x - (\sigma/3)\beta \ . \tag{A18}$$

Now, let

$$V_t = cV_{xx} + dV_x + eV \ . \tag{A19}$$

Then, substituting for (θ, ϕ) [using (A9)–(A11) and (A14)–(A18)] in Eq. (A3) and requiring that $V_{txxx} = V_{xxxt}$, gives us

$$c = 3(a_x + \sigma) \ , \qquad d = -a_{xx} - a^2 \ , \qquad e = -2\sigma a \ , \tag{A20}$$

and

$$a_t + \frac{\partial}{\partial x}\left(a_{xxxx} - 5aa_{xx} + \frac{5}{3}a^3\right) \ . \tag{A21}$$

Equation (A21) is the CDG equation (A2) with the identification

$$a = -6U \tag{A22}$$

and Eqs. (A17) and (A19) are the Lax pair for Eq. (A21).

Finally, from equations (A6), (A9), and (A14),

$$a = -6U_2 + 6\frac{\partial^2}{\partial x^2}\ln V_2 \tag{A23}$$

[by (A1) and (A11)]. It is curious that, in the resummation used earlier [4] to calculate the Lax pair, ϕ was directly an eigenfunction, while here it is a ratio of eigenfunctions (A11).

By a further analysis it can be shown that, with (A14), (A15), (A17), and (A19),

$$\beta_t = 3(a_x - \sigma)\beta_{xx} - (a_{xx} + a^2)\beta_x + 2\sigma a\beta \ , \tag{A24}$$

which is Eq. (A19) with $\sigma \to -\sigma$. From (A18) and (A24), β is an eigenfunction of the Lax pair (with $\sigma \to -\sigma$).

Appendix B: Factorization of scalar operators and the Schwarzian derivative

It is well known that the Schrödinger operator

$$D^2 + \tfrac{1}{2}U = (D - \tfrac{1}{2}V)(D + \tfrac{1}{2}V) \ , \tag{B1}$$

where

$$U = V_x - \tfrac{1}{2}V^2 \ , \qquad D = \tfrac{\partial}{\partial x} \ , \tag{B2}$$

with

$$V = \phi_{xx}/\phi_x \ , \tag{B3}$$

$$U = \{\phi; x\} \ . \tag{B4}$$

From Eqs. (1.11) and (1.12), the third-order operator in the Lenard formula is

$$D^3 + 2UD + U_x = (D - V)D(D + V) \ , \tag{B5}$$

where (V, U) are defined by (B3) and (B4).

We claim that the operator sequence

$$L_{n+1} = \prod_{j=0}^{n} \left[D + \left(j - \frac{n}{2} \right) V \right] = \left(D - \frac{n}{2} V \right) \cdots \left(D + \frac{n}{2} V \right) \ , \tag{B6}$$

where $n = 0, 1, 2, 3, \ldots$, and

$$V = \phi_{xx}/\phi_x \ , \tag{B7}$$

defines operators whose coefficients, when expanded, can be expressed entirely in terms of the Schwarzian derivative

$$U = V_x - \tfrac{1}{2}V^2 = \{\phi; x\} \tag{B8}$$

and its derivatives of order $(N-2)$ or less. From (B6)

$$\deg(L_n) = n \tag{B9}$$

and the operator (B1) is L_2 and (BS) is L_3.

The proof of the above uses the following result of Lavie [31]: A differential expression that is invariant under the Möbius group

$$\phi = (a\psi + b)/(c\psi + d) \tag{B10}$$

is a functional of the Schwarzian derivative. In particular, a polynomial differential invariant of order n is a polynomial in the Schwarzian derivative and suitable derivatives of the Schwarzian derivatives of order $n-3$ or less.

The Möbius transformation (B10) can be expressed as the composition of three operations (i) scaling

$$\phi = a\psi \ , \tag{B11}$$

(ii) translation

$$\phi = \psi + b \ , \tag{B12}$$

(iii) inversion

$$\phi = -1/\psi \ . \tag{B13}$$

From (B7) the coefficients of (B6) are invariant under (B11) and (B12). Therefore, it remains to show that (B6) is invariant under (B13).

Now

$$D + mV = D + m(\phi_{xx}/\phi_x) = \phi_x^{-m} D \phi_x^m \ , \tag{B14}$$

and from (B14)

$$L_{n+1} = \phi_x^{n/2+1} (\phi_x^{-1} D)^{n+1} \phi_x^{n/2} \ . \tag{B15}$$

Under the change of variable

$$\phi \to x \ , \qquad x \to \phi \ , \tag{B16}$$

$$\phi_x^{-1} = x_\phi \ , \qquad \phi_x^{-1} D = \frac{\partial}{\partial \phi} = D_\phi \ . \tag{B17}$$

The operator

$$L_{n+1} = x_\phi^{-n/2-1} D_\phi^{n+1} x_\phi^{-n/2} \ . \tag{B18}$$

With the change of variable (B13)

$$D_\phi = \psi^2 D_\psi \ , \tag{B19}$$

and

$$L_{n+1} = x_\psi^{-n/2-1} \psi^{-n-2} (\psi^2 D_\psi)^{n+1} \psi^{-n} x_\psi^{-n/2} \ . \tag{B20}$$

Now

$$\psi^{-n-2} (\psi^2 D_\psi)^{n+1} \psi^{-n} = \psi^{-n} \{ D_\psi \psi^2 D_\psi \psi_\psi^2 \cdots \psi^2 D_\psi \} \psi^{-n} \ . \tag{B21}$$

We will show that

$$\text{(B21)} = D_\psi^{n+1} \ , \tag{B22}$$

demonstrating that (B18) [and (B15)] is invariant under (B13) and, by the previous remark, under (B10). Equation (B22) is trivial when $n = 0$. When $n = 1$, the identity

$$D_\psi \psi^2 D_\psi = \psi D_\psi^2 \psi \tag{B23}$$

demonstrates (B22).

Now, by induction on n, (B22) is equivalent to

$$\psi^n D_\psi^{n+1} \psi^n = D_\psi \psi^n D_\psi^{n-1} \psi^n D_\psi \ , \tag{B24}$$

and directly

$$D_\psi \psi^n D_\psi^{n-1} \psi^n D_\psi = \psi^n D_\psi^{n+1} \psi^n + nR_n \ , \tag{B25}$$

where

$$R_n = \psi^{n-1} D_\psi^{n-1} \psi^n D_\psi - \psi^n D_\psi^n \psi^{n-1} \ . \tag{B26}$$

Now

$$R_n = \psi^{n-1}\{D_\psi^n \psi - \psi D_\psi^n - nD_\psi^{n-1}\}\psi^{n-1} \ , \tag{B27}$$

and the identity

$$D_\psi^n \psi = \psi D_\psi^n + nD_\psi^{n-1} \tag{B28}$$

establishes that

$$R_n = 0$$

and (B24) and (B22).

Therefore, by the result of Lavie, the operators have coefficients that are polynomial in the Schwarzian derivative (and its derivatives) when

$$V = \phi_{xx}/\phi_x \ .$$

For reference, the first few operators are

$$\begin{aligned}
L_1 &= D \ , \\
L_2 &= D^2 + \tfrac{1}{2}U \ , \\
L_3 &= D^3 + 2UD + U_x \ , \\
L_4 &= D^4 + 5UD^2 + 5U_xD + \tfrac{3}{2}(U_{xx} + \tfrac{3}{2}U^2) \ , \\
L_5 &= D^5 + 10UD^3 + 15U_xD^2 + 9(U_{xx} + 16U^2)D \\
&\quad + 2\frac{\partial}{\partial x}(U_{xx} + 4U^2) \ , \\
L_6 &= D^6 + \tfrac{35}{2}UD^4 + 35U_xD^3 + (\tfrac{63}{2}U_{xx} + \tfrac{259}{4}U^2)D^2 \\
&\quad + \frac{\partial}{\partial x}(14U_{xx} + \tfrac{259}{4}U^2)D + U_{xxxx} + \tfrac{31}{2}UU_{xx} + 13U_x^2 + \tfrac{45}{4}U^3 \ , \\
L_7 &= D^7 + 28UD^5 - 20U_xD^4 + (84U_{xx} + 196U^2)D^3 \\
&\quad + (56U_{xxx} + 588UU_x)D^2 + (20U_{xxxx} + 352UU_{xx} + 295U_x^2 \\
&\quad + 288U^3)D + \frac{\partial}{\partial x}(U_{xxxx} + 26UU_{xx} + \tfrac{33}{2}U_x^2 + 48U^3) \ ,
\end{aligned} \tag{B29}$$

where

$$U = V_x - \tfrac{1}{2}V^2 = \{\phi; x\} \ . \tag{B30}$$

Note that under the scaling

$$x \to a^{-1}x \ , \tag{B31}$$

from (B6) and (B7)

$$L_{n+1} \to a^{n+1}L_{n+1} \ , \tag{B32}$$

and each term of the expanded operator bas the same weight, where in accord with (B30),

$$U \to a^2 U \ . \tag{B33}$$

We also note the simplicity of the operator forms (B18).

Remark (1): The operators (L_1, L_3) define the dual Hamiltonian structure of the KdV equation.

Remark (2): The operator (L_2, L_3) do not define the Lax pair for the KdV equation.

Remark (3): From (B6), the coefficient of D^{n-1} in the expansion for L_n vanishes.

Remark (4): The L_{2k} operators are symmetric, the L_{2k+1} are antisymmetric.

Remark (5): For the CDG sequence the defining operators [Eqs. (1.22) and (1.23)] are

$$\begin{aligned} \theta_1 &= \theta_2 = L_3 \ , \\ J_1 &= D^{-1}L_2DL_2D^{-1} \ , \\ J_2 &= D^{-1}L_5D^{-1} \ . \end{aligned} \tag{B34}$$

Now consider the N-component, two-dimensional, periodic Toda lattice [14]

$$\theta_{jxt} = e^{\theta_j - \theta_{j-1}} - e^{\theta_{j+1}-\theta_j} \ , \tag{B35}$$

where $j = 1, 2, 3, \ldots \mod(N)$. The N^{th} order, scalar operator

$$L = \prod_{j=0}^{n} (D - \theta_{jx}) \ , \tag{B36}$$

where $\sum_1^n \theta_{jx} = 0$, is the scattering part of the Lax pair for (B35). It is not difficult to see that (1) when (B36) is L_2, Eq. (B35) reduces to the sine-Gordon system; (2) when (B36) is L_3, Eq. (B35) reduces to the Bullough-Dodd Eq. (2.7); (3) the identification of (B36) with L_2, for $n > 2$ does not lead to a consistent scalar reduction of (B35). Therefore, subject to the identification of (B36) with (B6), the sine-Gordon and Bullough-Dodd equations represent the only consistent scalar reductions of Eqs. (B35). If the sequence (B6) is unique (in the sense that L_n is the unique factorization of an n^{th} order scalar operator consistent with the Schwarzian formulation and not a product of operators of lower degrees) then the results of Sec. I show that, within a wide class of equations, the KdV and CDG sequences are the unique instances of scalar evolution equations (See [4] and especially [32]).

Remark (6): It is easy to see, from (B6) and (B7), that a basis for the null space of the operator L_{n+1} is

$$\{\phi_x^{-n/2}\phi^j \ ; \quad j = 0, 1, 2, \ldots, n\} \ . \tag{B37}$$

Therefore, the general null function, which satisfies

$$L_{Ln+1} f_n = 0 \ , \tag{B38}$$

is

$$f_n = \phi_x^{-n/2} \sum_{j=0}^{n} c_j \phi^j \ , \tag{B39}$$

where the $\{c_j\}$ are constants.

In a previous work [4] we have investigated the class of partial differential equations

$$\phi_t/\phi_x + B(\{\phi; x\}) = 0 \tag{B40}$$

that are formulated in terms of the Schwarzian derivative and have a transformation

$$\phi_x = \psi_x^m \ , \tag{B41}$$

which preserves the formulation (B40) in terms of the Schwarzian derivative. The KdV sequence corresponds to $m = -1$ and the CDG sequence corresponds to $m = -2$.

In terms of the "modified" variables

$$V = \phi_{xx}/\phi_x \ , \qquad W = \psi_{xx}/\psi_x \ , \tag{B42}$$

the symmetry (B41) corresponds to

$$V = mW \ . \tag{B43}$$

In analogy with the recursion operators for the modified KdV and CDG sequence define the "recursion operators"

$$\Omega_{n+2} = D(D + V)D^{-1} \circ L_{n+1} \circ D^{-1}(D - V) \ , \tag{B44}$$

where

$$V = \phi_{xx}/\phi_x \ . \tag{B45}$$

The "sequence" of modified equations are

$$V_t + \Omega_{n+2}^j \circ V_x = 0 \ , \tag{B46}$$

which corresponds to equations of the form (B40), where

$$\phi_t/\phi_x + M_{n+2}^j \circ 1 = 0 \ , \tag{B47}$$

where

$$\begin{aligned} M_{n+2} &= D^{-1} \circ L_{n+1} \circ D^{-1} \circ \theta \ , \\ \theta &= (D - V)D(D + V) = L_3 \ . \end{aligned} \tag{B48}$$

We note that Ω_4 is the modified KdV recursion operator and Ω_6 is the modified CDG operator. (Actually, $\Omega_4 = \Omega_2^2$, where Ω_2 is the MKdV recursion operator). From the identity

$$(D + aV)D^{-1}(D + bV) = (D + bV)D^{-1}(D + aV) \ , \tag{B49}$$

the definition of L_{n+1} (B6), and the form of the Eqs. (B46), a symmetry of the form (B43) that preserves the Schwarzian formulation of (B46) and (B47) requires that

$$m = -n/2 \ . \tag{B50}$$

Again, if the operators L_n or the composite operators

$$\prod_j L_{kj} \ , \tag{B51}$$

where $\sum_j k_j = n$ are the unique operators of order n with a product form and Schwarzian coefficients, then the symmetry (B43) is allowed only if $n = 2$ or $n = 4$. That is,

$$m = -1, -2 \ , \tag{B52}$$

which are the KdV and CDG sequences.

It may be of some interest to note that the "Harry Dym" sequence corresponding to (B47) is

$$x_t = M_{n+2}^j \circ 1 \ , \tag{B53}$$

where

$$M_{n+2} = D_\phi^{-1} x_\phi^{-n/2} D_\phi^{n+1} x_\phi^{-n/2} D_\phi^{-1} x_\phi^{-1} D_\phi^3 x_\phi^{-1} \ .$$

Finally, it is not difficult to show that if Eq. (B40) has a transformation (B41), which preserves the Schwarzian formulation, then the resulting equations will have the Painlevé property only for the index pairs $(m, 1/m)$:

$$\text{(i)} \quad (-1,-1) \ , \qquad \text{(ii)} \quad (-2,-\tfrac{1}{2}) \ . \tag{B54}$$

That is, only for the KdV and CDG sequences. To see this, we start with the "generic" singularity:

$$\varphi = \varphi_0/\varepsilon + \varphi_1 + \cdots \ , \tag{$B55$}$$

which is single-valued for Eqs. (B40) (see [4]). Then, by applying (B41) to (B55) we obtain the result.

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