Modified equations, rational solutions, and the Painlevé property for the Kadomtsev-Petviashvili and Hirota-Satsuma equations

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Abstract

We propose a method for finding the Lax pairs and rational solutions of integrable partial differential equations. That is, when an equation possesses the Painlevé property, a Bäcklund transformation is defined in terms of an expansion about the singular manifold. This Bäcklund transformation obtains (1) a type of modified equation that is formulated in terms of Schwarzian derivatives and (2) a Miura transformation from the modified to the original equation. By linearizing the (Ricati-type) Miura transformation the Lax pair is found. On the other hand, consideration of the (distinct) Bäcklund transformations of the modified equations provides a method for the iterative construction of rational solutions. This also obtains the Lax pairs for the modified equations. In this paper we apply this method to the Kadomtsev-Petviashvili equation and the Hirota-Satsuma equations.

1 Introduction

In [1] we have formulated a procedure for calculating the Lax pair and rational solutions of partial differential equations that possess the Painlevé property. That is, for an equation with the Painlevé property, a Bäcklund transformation is defined in terms of an expansion about the "singular manifold". This Bäcklund transformation obtains (1) a type of "modified equation" that can be expressed in terms of Schwarzian derivatives and (2) a Miura transformation from the modified to the original equation. By linearizing the Ricati-type Miura transformation (and the modified equations), the Lax pair is found. Then, further consideration of the Bäcklund transformations for the modified equations provides a method for the iterative construction of "rational" solutions, and finds the Lax pair for the modified equations as well.

We recall that the partial differential equation is said to possess the Painlevé property [2]–[7] when the solutions of the partial differential equation (pde) are "single valued" about the movable, singularity manifold and the singularity manifold is "noncharacteristic." To be precise, if the singularity manifold is determined by

$$\varphi(z_1, z_2, \dots, z_n) = 0 \ , \tag{1.1}$$

and $u = u(z_1, \dots, z_n)$ is a solution of the pde, then we require that

$$u = \varphi^\alpha \sum_{j=0}^{\infty} u_j \varphi^u \ , \tag{1.2}$$

where $u_0 \neq 0$, $\varphi = \varphi(z_1, \dots, z_n)$, and $u_j = u_j(z_1, \dots, z_n)$ are analytic functions of (z_j) in a neighborhood of the manifold (1.1) and α (the leading-order exponent) is a (negative) rational number. The requirement that the manifold (1.1) be noncharacteristic (for the pde) insures that the expansion (1.2) will be well defined, in the sense of the Cauchy-Kovalevskaya theorem [8]. Substitution of(1.2) into the pde determines that value(s) of α, and defines the recursion relations for u_j, $j = 0, 1, 2, \dots$. When the expansion (1.2) is well defined and contains the maximum number of arbitrary functions allowed at the "resonances" [2], [9], [10], the pde is said to possess the Painlevé property and is conjectured to be integrable. Informally, the resonances are the values of j for which the u_j are not "fixed" by the recursion relations (i.e., are arbitrary).

The Bäcklund transformation is defined by truncating the expansion (1.2) at the constant level term. That is, we set

$$u = u_0 \varphi^{-n} + u_1 \varphi^{-n+1} + \cdots + u_n \ , \tag{1.3}$$

and find, from the recursion relations for u_j and the condition that u_j vanish for $j > n$, a system of equations for $(\varphi,\ u_j,\ j = j0, 1, \dots, n))$, where u_n will satisfy the (original) pde. This system of equations will, in general (depending on the values of the resonances), be overdetermined. Upon solving this system, it is found, for those equations considered, the φ satisfies an equation formulated in terms of Schwarzian derivatives [3]:

$$\{\varphi; x\} = \frac{\partial}{\partial x}\left(\frac{\varphi_{xx}}{\varphi_x}\right) - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \ . \tag{1.4}$$

This equation, or system of equations, we regard as a type of modified equation. By the invariance of (1.4) under the Moebius group,

$$\varphi = (a\psi + b)/(c\psi + d) \ , \qquad \{\varphi; x\} = \{\psi; x\} \ , \tag{1.5}$$

the "modified" equations allow the Bäcklund transformation (1.5).

The above procedure may now be reapplied to the "modified" (or equivalent) equations to find different forms of Bäcklund transformations. These Bäcklund transformations may take the form of discrete symmetries [1],[5], [6], reductions [1], or, as we shall see, more complicated structures. The group of Bäcklund transformations for the modified equations may be conveniently employed to iteratively construct sequences of rational solutions. Also, by linearizing the Miura transformation from modified to original equation we propose to calculate the Lax pair [1], [6].

In this paper we consider the Kadomtsev-Petviashvili (KP) equation and the Hirota-Satsuma equations. The modified equations are derived, their (modified) Bäcklund transformations are calculated, and the sequences of rational solutions are found.

2 The Kadomtsev-Petviashvili Equation

The Kadomtsev-Petviashvili equation

$$U_{yy} + \frac{\partial}{\partial x}(U_t + UU_x + U_{xxx}) = 0 \tag{2.1}$$

possesses the Painlevé property [2]. The expansion about the singular manifold ($\varphi = 0$) is

$$U = \varphi^{-2} \sum_{j=0}^{\infty} U_j \varphi^j \ . \tag{2.2}$$

with resonances at

$$j = -1, 4, 5, 6 \ . \tag{2.3}$$

Therefore, subject to the "noncharacteristic" condition ($\varphi_x \neq 0$ when $\varphi = 0$), $\{\varphi, U_4, U_5, U_6\}$ are arbitrary functions of (x, t) in the expansion (2.2).

The Bäcklund transformation [2], [3] for Eq. (2.1) is

$$U = U_0 \varphi^{-2} + U_1 \varphi^{-1} + U_2 \ , \tag{2.4}$$

which obtains

$$
\begin{aligned}
&U_0 = -12\varphi_x^2 \ , \qquad U_1 = 12\varphi_{xx} \ , \\
&U_2 + \frac{\varphi_t}{\varphi_x} + 4\frac{\varphi_{xxx}}{\varphi_x} - 3\frac{\varphi_{xx}^2}{\varphi_x^2} + \frac{\varphi_y^2}{\varphi_x^2} = 0 \ , \\
&\varphi_{xt} + \varphi_{xxxx} + \varphi_{yy} + \varphi_{xx}u_2 = 0 \ .
\end{aligned}
\tag{2.5}
$$

We note that the system (2.5) is not overdetermined since (U_4, U_5, U_6) may vanish without restriction. From (2.4) and (2.5) it is found that

$$
U = 12\frac{\partial^2}{\partial x^2}\ln\varphi + U_2 \ , \tag{2.6}
$$

$$
U_2 + \frac{\varphi_t}{\varphi_x} + 4\frac{\varphi_{xxx}}{\varphi_x} - 3\frac{\varphi_{xx}^2}{\varphi_x^2} + \frac{\varphi_y^2}{\varphi_x^2} = 0 \ , \tag{2.7}
$$

and

$$
\frac{\partial}{\partial y}\left(\frac{\varphi_y}{\varphi_x}\right) + \frac{\partial}{\partial x}\left(\frac{\varphi_t}{\varphi_x} + \{\varphi; x\} + \frac{1}{2}\frac{\varphi_y^2}{\varphi_x^2}\right) = 0 \ , \tag{2.8}
$$

where

$$
\{\varphi; x\} = \frac{\partial}{\partial x}\left(\frac{\varphi_{xx}}{\varphi_x}\right) - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \ . \tag{2.9}
$$

In terms of our procedure, Eq. (2.8) is the “modified” equation formulated in terms of the Schwarzian derivative (2.9) and Eq. (2.7) is a “Miura” transformation from Eq. (2.8) to Eq. (2.1). Equation (2.8) is invariant under the Moebius group

$$
\varphi = (a\psi + b)/(c\psi + d) \ , \tag{2.10}
$$

where $ad - bc \neq 0$.

To investigate the group of Bäcklund transformations for Eq. (2.8), it is convenient to study various forms of “modified” equations that are equivalent to (2.8). To begin we let

$$
\frac{V = \varphi_{xx}/\varphi_x \ , \qquad W = \varphi_y}{\varphi_x \ , \qquad Z = \varphi_t/\varphi_x \ ,} \tag{2.11}
$$

and find, from (2.8) and (2.11), the system of modified equations

$$
\begin{aligned}
&W_y + \frac{\partial}{\partial x}\left(Z + \frac{W^2}{2} + V_x - \frac{1}{2}V^2\right) = 0 \ , \\
&V_y = \frac{\partial}{\partial x}(W_x + VW) \ , \qquad V_t = \frac{\partial}{\partial x}(Z_x + vZ) \ ,
\end{aligned}
\tag{2.12}
$$

where

$$W_t + WZ_x = Z_y + ZW_x \ . \tag{2.13}$$

Equations (2.12) and (2.13) are overdetermined. Equation (2.13) arises from the condition $V_{yt} = V_{ty}$. This system allows singularities of the form

$$V \sim V_0 \epsilon^\alpha \ , \qquad W \sim \omega_0 \epsilon^\beta \ , \qquad Z \sim Z_0 \epsilon^\gamma \ , \tag{2.14}$$

where

(i) $\alpha = -1 \ , \qquad \beta = \gamma = 0 \ , \qquad V_0 = 0 - 2\epsilon_x \ ;$ (2.15)

(ii) $\alpha = \beta = \gamma = -1 \ , \qquad V_0 = \epsilon_x \ , \qquad \omega_0^2 = 3\epsilon_x^2 \ ;$ (2.16)

(iii) $\alpha = -1 \ , \qquad \beta = -2 \ , \qquad \gamma = 0 \ ,$ (2.17)

$V_0 = 2\epsilon_x \ , \qquad \omega_0 = \epsilon_y \ , \qquad Z_0 = 4\epsilon_x \ .$

For (2.16) and (2.17) the resonances are

$$j = -1, 0, 2, 2, 2, 3 \tag{2.18}$$

and

$$j = -2, -1, 1, 2, 3, 4 \ , \tag{2.19}$$

respectively. The expansion about the singularity (2.16) contains the arbitrary functions $(\epsilon, Z_0, V_2, W_2, Z_2, \{V_3, W_3\})$. The Bäcklund transformation is

$$\begin{aligned} &V = v_0 \epsilon^{-1} + V_1 \ , \qquad W = W_0 \epsilon^{-1} + W_1 \ , \\ &Z = Z_0 \epsilon^{-1} + Z_1 \ , \end{aligned} \tag{2.20}$$

where

$$V_0 = \epsilon_x \ , \qquad W_0 = a\epsilon_x \ , \qquad a^2 = 3 \ , \qquad Z_0 = H\epsilon_x \ , \tag{2.21}$$

$$\begin{aligned} V_1 &= \frac{1}{2}\left(-\frac{\epsilon_{xx}}{\epsilon_x} + a\frac{\epsilon_y}{\epsilon_x} + \frac{H}{2}\right) \ , \\ W_1 &= \frac{1}{2}\left(-a\frac{\epsilon_{xx}}{\epsilon_x} - \frac{\epsilon_y}{\epsilon_x} - \frac{aH}{2}\right) \ , \end{aligned} \tag{2.22}$$

$$\begin{aligned} &Z_1 \frac{\epsilon_t}{\epsilon_x} - H - x - \frac{H^2}{4} - \frac{1}{2}\frac{\epsilon_{xx}}{\epsilon_x} H - \frac{a}{2}\frac{\epsilon_y}{\epsilon_x} H \ , \\ &H_y = \frac{\partial}{\partial x}\left(a\left(H_x + \frac{H^2}{4}\right) + 2\frac{\epsilon_y}{\epsilon_x} H - 2a\frac{\epsilon_t}{\epsilon_x}\right) = 0 \ , \end{aligned} \tag{2.23}$$

and

$$\frac{\partial}{\partial y}\left(\frac{\epsilon_y}{\epsilon_x}\right) + \frac{\partial}{\partial x}\left(\frac{\epsilon_t}{\epsilon_x} + \{\epsilon; x\} + \frac{1}{2}\left(\frac{\epsilon_y}{\epsilon_x}\right)^2\right) = 0 \ . \tag{2.24}$$

To simplify the above let

$$V = \epsilon_{xx}/\epsilon_x \ , \qquad W = \epsilon_y/\epsilon_x \ , \qquad Z = \epsilon_t/\epsilon_x \ , \tag{2.25}$$

and find

$$\begin{aligned}
&\text{(i)} \qquad a(V + V_1) = W - W_1 \ , \\
&\text{(ii)} \qquad V_1 - V = H + a(W + W_1) \ , \\
&\text{(iii)} \qquad Z - Z_1 = H_x + (V + V_1)H \ ,
\end{aligned} \tag{2.26}$$

where the auxiliary function H satisfies (2.23). Now (2.26) constitutes a somewhat awkward Bäcklund transformation (BT) for Eqs. (2.12) and (2.13), which can be simplified by identifying (2.12) with (2.8) through

$$\begin{aligned}
V &= \psi_{xx}/\psi_x \ , \qquad V_1 = \varphi_{xx}/\varphi_x \ , \\
W &= \psi_y/\psi_x \ , \qquad W_1 = \varphi_y/\varphi_x \ , \\
Z &= \psi_t/\psi_x \ , \qquad Z_1 = \varphi_t/\varphi_x \ .
\end{aligned} \tag{2.27}$$

Thus, after simplification, eliminating H in Eq. (2.26) obtains the BT

$$\psi_y = a\psi_{xx} + A\psi_x \ , \qquad \psi_t = -4\psi_{xxx} - 2aA\psi_{xx} + B\psi_x \ , \tag{2.28}$$

where (ψ, φ) satisfy (2.8) and

$$\begin{aligned}
&a^2 = 3 \ , \qquad A = \varphi_y/\varphi_x + a(\varphi_{xx}/\varphi_x) \ , \\
&B = \frac{\varphi_t}{\varphi_x} - 2\left(\frac{\varphi_{xxx}}{\varphi_x} + a\frac{\varphi_{xy}}{\varphi_x}\right) \ .
\end{aligned} \tag{2.29}$$

That $(\psi\varphi)$ satisfy (2.8) is found from the conditions $(\varphi_{yt} = \varphi_{ty},\ \psi_{yt} = \psi_{ty})$, respectively. Having found (2.28) we discontinue consideration of the system (2.12) and instead consider the system in (A, B) obtained from (2.28) by the condition, $\psi_{yt} = \psi_{ty}$,

$$\begin{aligned}
&A_y + \psi(aA_x + \tfrac{1}{2}A^2 + B) = 0 \ , \\
&B_y + BA_x - 4A_{xxx} - 2aAA_{xx} = aB_{xx} + AB_x + A_t \ .
\end{aligned} \tag{2.30}$$

Now, the expression

$$\Omega = B + A^2 = \frac{\varphi_t}{\varphi_x} - 2\{\varphi; x\} - 2a\frac{\partial}{\partial x}\left(\frac{\varphi_y}{\varphi_x}\right) + \left(\frac{\varphi_y}{\varphi_x}\right)^2 \tag{2.31}$$

is invariant under (2.10). This suggests defining the system in (A, Ω):

$$
\begin{aligned}
&A_y + \frac{\partial}{\partial x}\left(aA_x - \frac{A^2}{2} + \Omega\right) = 0 \ , \\
&\Omega_y - A_t = \frac{\partial}{\partial x}\left(4A_{xx} + \frac{A^3}{3} - 2aAA_x + a\Omega_x - 4\Omega\right) \ .
\end{aligned}
\tag{2.32}
$$

Note that Eqs. (2.30) or (2.32) are "properly posed" in comparison to Eqs. (2.12) in that they are not overdetermined and have the same order as Eq. (2.1) or Eq. (2.8). From the Miura transformation (2.7) and the above,

$$
\begin{aligned}
-U_2 &= \frac{\varphi_t}{\varphi_x} + 4\frac{\varphi_{xxx}}{\varphi_x} - 3\frac{\varphi_{xx}^2}{\varphi_x^2} + \frac{\varphi_y^2}{\varphi_x^2} \\
&= B + 2aA_x + A^2 = \Omega + 2aA_x \ .
\end{aligned}
\tag{2.33}
$$

By construction (2.28) constitute a Lax pair for Eqs. (2.30). The Lax pair for (2.32) is found by substituting for B in (2.30) using (2.31).

Equations (2.32) allow two Bäcklund transformations:

(i) $$A = 2a\frac{\psi_x}{\psi} + A_1 \ , \qquad \Omega = -12\frac{\partial^2}{\partial x^2}\ln\psi + \Omega_2 \ , \tag{2.34}$$

where

$$A_1 = \frac{\psi_y}{\psi_x} - a\frac{\psi_{xx}}{\psi_x} \ , \qquad \Omega_2 = \frac{\psi_t}{\psi_x} + \frac{\psi_y^2}{\psi_x^2} + 4\frac{\psi_{xxx}}{\psi_x} - 3\frac{\psi_{xx}^2}{\psi_x^2} \ , \tag{2.35}$$

and ψ satisfies (2.8).

(ii) $$A = -2a(\varphi_x/\varphi) + A_1 \ , \qquad \Omega = \Omega_2 \ , \tag{2.36}$$

where φ satisfies (2.8) and

$$
\begin{aligned}
A_1 &= \frac{\varphi_y}{\varphi_x} + a\frac{\varphi_{xx}}{\varphi_x} \ , \\
\Omega &= \frac{\varphi_t}{\varphi_x} - 2\{\varphi; x\} - 2a\frac{\partial}{\partial x}\left(\frac{\varphi_y}{\varphi_x}\right) + \left(\frac{\varphi_y}{\varphi_x}\right)^2 \ .
\end{aligned}
\tag{2.37}
$$

Equations (2.35) re obtain the Lax pair for (2.32), where the identification (2.33),

$$-U_2 = \Omega_2 \ . \tag{2.38}$$

Equations (2.37) provide (2.29) and, with (2.35), obtain (2.28).

Now consider the BT/Lax pair (2.28). We have the following:

Lemma. For fixed (A, B) let (ψ_1, ψ_2) be two linearly independent solutions of Eqs. (2.28), then

$$\psi = \psi_{1x}/\psi_{2x} \tag{2.39}$$

will satisfy Eqs. (2.28) with

$$A \to A' \ , \qquad B \to B' \ ,$$

where (A, B), (A', B') satisfy Eqs. (2.30) and

$$\begin{aligned} A' &= A + 2a(\psi_{2xx}/\psi_{2x}) \ , \\ B' &= B - 2aA'_x - A'^2 + A^2 \\ &= B - 2aA_x - \frac{\psi_{2xx}}{\psi_{2x}}A - 12\frac{\psi_{2xxx}}{\psi_{2x}} \ . \end{aligned} \tag{2.40}$$

Proof. By direct calculation, which we omit.

To investigate the iterative application of the BT (2.28), we define a double sequence

$$\begin{aligned} \varphi_{j+1,y} &= a\varphi_{j+1,xx} + A_j\varphi_{j+1,x} \ , \\ \psi_{j+1,y} &= a\psi_{j+1,xx} + A_j\psi_{j+1,x} \ , \end{aligned} \tag{2.41}$$

$$\begin{aligned} \varphi_{j+1,t} &= -4\varphi_{j+1,xxx} - 2aA_j\varphi_{j+1,xx} + B_j\varphi_{j+1,x} \ , \\ \psi_{j+1,t} &= -4\psi_{j+1,xx} - 2aA_j\psi_{j+1,xx} + B_j\psi_{j+1,x} \ , \end{aligned} \tag{2.42}$$

where

$$A_j = \frac{\varphi_{j,y}}{\varphi_{j,x}} + a\frac{\varphi_{j,xx}}{\varphi_{j,x}} = A_{j-1} + 2a\frac{\varphi_{j,xx}}{\varphi_{j,x}} \ , \tag{2.43}$$

$$B_j = \frac{\varphi_{j,t}}{\varphi_{j,x}} - 2\left(\frac{\varphi_{j,xx}}{\varphi_{j,x}} + a\frac{\varphi_{j,xx}}{\varphi_{j,x}}\right) \ , \tag{2.44}$$

$$B_j = B_{j-1} - 2aA_{j,x} - A_j^2 + A_{j-1}^2 \ . \tag{2.45}$$

Then, by the lemma, it is consistent to set

$$\varphi_{j+1} = \psi_{j,x}/\varphi_{j,x} \ . \tag{2.46}$$

Now let

$$A_0 = B_0 = 0 \tag{2.47}$$

or

$$\varphi_{1y} = a\varphi_{1xx} \ , \qquad \varphi_{1t} = -4\varphi_{1xxx} \ . \tag{2.48}$$

By (2.43) and (2.45),

$$A_j = 2a \sum_{k=1}^{j} \frac{\varphi_{k,xx}}{\varphi_{k,x}} \ , \qquad B_j = -2a \frac{\partial}{\partial x} \sum_{\ell=1}^{j} A_\ell - A_j^2 \ , \tag{2.49}$$

and by (2.33), for $j > 1$,

$$\begin{aligned} U_{2j} &= -B_j - 2aA_{j,x} - A_j^2 = 2a \frac{\partial}{\partial x} \sum_{\ell=1}^{j-1} A_\ell \ , \\ U_{2j} &= 12 \frac{\partial^2}{\partial x^2} \ln \prod_{1}^{j-1} \varphi_{\ell,x}^{(j-\ell)} \ , \end{aligned} \tag{2.50}$$

$$U_j = 12 \frac{\partial^2}{\partial x^2} \ln \varphi_j + U_{2j} = 12 \frac{\partial^2}{\partial x^2} \ln \left\{ \psi_{j-1,x} \prod_{1}^{j-2} \varphi_{\ell,x}^{(j-\ell)} \right\} \ . \tag{2.51}$$

In effect, to iterate (2.28) two solutions are "interpolated" to produce one new solution at each step. From N linearly independent solutions at one level, fixing one solution as the denominator in (2.46) produces $N-1$ solutions of (2.41) and (2.42) at the next level. However, from the linearity of (2.48), it is possible to generate an infinity of linearly independent solutions. For instance, to find rational solutions, let

$$\varphi_y = a\varphi_{xx} \ , \qquad \varphi_t = -4\varphi_{xxx} \ , \tag{2.52}$$

and

$$\varphi^{(2n)} = \sum_{j=0}^{n} b_j x^j \tag{2.53}$$

where

$$\begin{aligned} &b_{j-2,y} = aj(j-1)b_j \ , \qquad b_{j-3,t} = -4j(j-1)(j-2)b_j \ , \\ &ab_{j-1,t} = -4jb_{j,y} \ . \end{aligned} \tag{2.54}$$

This obtains

$$\begin{aligned} &\varphi_1^0 = 1 \ , \qquad \varphi_1^1 = x \ , \qquad \varphi_1^2 = x^2 + 2ay \ , \\ &\varphi_1^3 = x^3 + 6ayx - 24t \ , \\ &\varphi_1^4 = x^4 + 12ayx^2 - 96tx + 36y^2 \ , \\ &\varphi_1^5 = x^5 + 20ayx^3 - 960tx^2 + 180y^2x - 288ayt \ . \end{aligned} \tag{2.55}$$

Using the identity

$$j\varphi_x^{(j)} = \varphi^{(j-1)} \ , \tag{2.56}$$

and letting

$$\varphi_1 = \varphi_1^5 \ ,$$

a first application of (2.41), (2.42), and (2.46) finds the appropriate set of solutions at the next level. That is,

$$\begin{aligned} \varphi_2^1 &= 1/\varphi_1^4 \ , \qquad \varphi_2^2 = \varphi_1^1/\varphi_1^4 \ , \\ \varphi_2^3 &= \varphi_1^2/\varphi_1^4 \ , \qquad \varphi_2^4 = \varphi_1^3/\varphi_1^4 \ . \end{aligned} \tag{2.57}$$

Then, with

$$\varphi_2 = \varphi_2^1 = 1/\varphi_1^4 \ , \tag{2.58}$$

Eq. (2.46) obtains

$$\begin{aligned} \varphi_3^1 &= 4\varphi_1^1 - \varphi_1^4/\varphi_1^3 \ , \qquad \varphi_3^2 = 4\varphi_1^2 - 2\varphi_1^1\varphi_1^4/\varphi_1^3 \ , \\ \varphi_3^3 &= 4\varphi_1^3 - 3\varphi_1^2\varphi_1^4/\varphi_1^3 \ , \end{aligned} \tag{2.59}$$

etc.

An identical procedure can be applied to *any* linearly independent set of solutions of Eqs. (2.52).

Finally, the KP equation (2.1) is invariant under the Galilean transformation

$$x' = x + \alpha y + \beta t \ , \qquad y' = y - 2\alpha t \ , \tag{2.60}$$

$$t' = t \ , \qquad u' = u - \alpha^2 - \beta \ . \tag{2.61}$$

Also, Eqs. (2.8), (2.30), and (2.32) are invariant under (2.60), where

$$A \to A + \alpha \ , \qquad B \to B - 2\alpha A + \beta \ , \qquad \Omega \to \Omega + \alpha^2 + \beta \ , \tag{2.62}$$

which is consistent with their definitions (2.29) and (2.31). Obviously, (2.60) preserves the form of the rational solutions and may be applied directly to, say, (2.55)–(2.59).

3 The Hirota-Satsuma Equations

The Hirota-Satsuma equations

$$u_t = \tfrac{1}{2}u_{xxx} + 3uu_x - 6\omega\omega_x \ , \qquad \omega_t = -\omega_{xxx} - 3u\omega_x \tag{3.1}$$

have the Painlevé property [6] about singularities of the form

(i) $$u = \psi^{-2} \sum_{j=0}^{\infty} u_j \psi^j \ , \qquad \omega = \psi^{-1} \sum_{j=0}^{\infty} \omega_j \psi^j \ , \tag{3.2}$$

with resonances

$$j = -1, 0, 1, 4, 5, 6 \ ; \tag{3.3}$$

(ii) $$u = \psi^{-2} \sum_{j=0}^{\infty} u_j \psi^j \ , \qquad \omega = \psi^{-2} \sum_{j=0}^{\infty} \omega_j \psi^j \ , \tag{3.4}$$

with resonances

$$j = -2, -1, 3, 4, 6, 8 \ . \tag{3.5}$$

The Bäcklund transformation about (3.2) is

$$u = 2 \frac{\partial^2}{\partial x^2} \ln \psi + u_2 \ , \qquad \omega = \frac{\omega_0}{\psi} + \omega_1 \ , \tag{3.6}$$

where, as found in [6]

$$\psi_t + \psi_{xxx} + 3\psi_x u_2 = 2\psi_x \vartheta \ , \tag{3.7}$$

$$\omega_0 = \psi_x H \ , \tag{3.8}$$

$$\psi_t/\psi_x - \tfrac{1}{2}\{\psi; x\} = \tfrac{3}{4} H^2 + \vartheta \ , \tag{3.9}$$

$$\vartheta_x^2 = (\lambda^2 + \vartheta^2) H^2 \ , \tag{3.10}$$

$$\omega_1 = -\tfrac{1}{2}\omega_{0x}/\psi_x - \tfrac{1}{3}(\lambda^2 + \vartheta^2)^{1/2} \ , \tag{3.11}$$

and

$$H_t + \frac{\partial}{\partial x}\left(H_{xx} + \frac{H^3}{4} + \vartheta H + \frac{3}{2}\{\psi; x\} H \right) = 0 \ . \tag{3.12}$$

In [6] we have found the Lax pair for (3.1) by "linearizing" the Miura transformation, (3.7) and (3.11), from the "modified equations" (3.9) and (3.12). To review, we let

$$W = \psi_{xx}/\psi_x \ , \tag{3.13}$$

and find the "modified" equations

$$W_t = \frac{1}{2}\frac{\partial}{\partial x}\left(W_{xx} - \frac{W^3}{2} + 3\left(H_x + \frac{WH}{2} \right) H + 2(\vartheta_x + W\vartheta) \right) \ , \tag{3.14}$$

$$H_t + \frac{\partial}{\partial x}\left(H_{xx} + \frac{1}{4}H^3 + \vartheta H + \frac{3}{2}\left(W_x - \frac{1}{2}W^2 \right) H \right) = 0 \ , \tag{3.15}$$

where

$$\vartheta_x^2 = (\lambda^2 + \vartheta^2) H^2 \ .$$

The Miura transformations are

$$\begin{aligned} -2u_2 &= W_x + \tfrac{1}{2}W^2 + \tfrac{1}{2}H^2 - \tfrac{2}{3}\vartheta \ , \\ -2\omega_1 &= H_x + WH + \tfrac{2}{3}(\lambda^2 + \vartheta^2)^{1/2} \ . \end{aligned} \tag{3.16}$$

Then letting

$$\begin{aligned} &W + H = 2(\epsilon_x/\epsilon) \ , \qquad W - H = 2(\beta_x/\beta) \ , \\ &\vartheta = \lambda \sinh \alpha \ , \qquad \alpha = \ln(\epsilon/\beta) \ , \end{aligned} \tag{3.17}$$

obtains the Lax pair from (3.14)–(3.16) (see [6]).

We now proceed to study the Bäcklund transformations of (3.14) and (3.15) when

$$\vartheta = \lambda = 0 \ . \tag{3.18}$$

The relevant equations are

$$\psi_t/\psi_x - \tfrac{1}{2}\{\psi; x\} = \tfrac{3}{4}H^2 \tag{3.19}$$

and

$$\begin{pmatrix} W \\ H \end{pmatrix}_t = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{W_{xx}}{2} - \frac{W^3}{4} + \frac{3}{2}\left(H_x + \frac{WH}{2}\right) H \\ -H_{xx} - \frac{H^3}{4} - \frac{3}{2}\left(W_x - \frac{W^2}{2}\right) H \end{pmatrix} \ . \tag{3.20}$$

Equations (3.20) allow the following singularities:

$$W \sim W_0 \varphi^{-1} \ , \qquad H \sim H_0 \varphi^{-1} \ , \tag{3.21}$$

(i) $\qquad W_0 = -2\varphi_x \ , \qquad H_0 = 0 \ ;$ (3.22)

(ii) $\qquad W_0 = 2\varphi_x \ , \qquad H_0 = 0 \ , \quad \pm 4\varphi_x \ ;$ (3.23)

(iii) $\qquad W_0 = \varphi_x \ , \quad -3\varphi_x \ , \qquad H_0^2 = \varphi_x^2 \ .$ (3.24)

When

$$W_0 = \varphi_x \ , \qquad H_0^2 = \varphi_x^2 \ , \tag{3.25}$$

the resonances are

$$j = -1, 1, 2, 3, 3, 4 \tag{3.26}$$

and the Bäcklund transformation is

$$W = \varphi_x/\varphi + W_1 \ , \qquad H = a(\varphi_x/\varphi) + H_1 \ , \tag{3.27}$$

where

$$a^2 = 1 \ , \tag{3.28}$$

$$aH_1 + W_1 = -\varphi_{xx}/\varphi_x \ , \tag{3.29}$$

$$\frac{\varphi_t}{\varphi_x} - \frac{1}{2}\{\varphi; x\} = \frac{3}{2}a\left(H_{1x} - \frac{\varphi_{xx}}{\varphi_x}H_1 - aH_1^2\right) \ , \tag{3.30}$$

and

$$H_{1t} + \frac{\partial}{\partial x}(H_{1xx} + \tfrac{1}{4}H_1^3 + \tfrac{3}{2}(W_{1x} - \tfrac{1}{2}W_1^2)H_1) = 0 \ . \tag{3.31}$$

We note that Eqs. (3.29)–(3.31) imply that (W_1, H_1) satisfy Eq. (3.20). Now, consistent with Eqs. (3.19) and (3.20), we define the variable ψ so that

$$W_1 = \psi_{xx}/\psi_x \ , \tag{3.32}$$

$$\psi_t/\psi_x - \tfrac{1}{2}\{\psi; x\} = \tfrac{3}{4}H_1^2 \ , \tag{3.33}$$

$$H_{1t} + \frac{\partial}{\partial x}\left(H_{1xx} + \frac{H_1^3}{4} + \frac{3}{2}\{\psi; x\}H_1\right) = 0 \ . \tag{3.34}$$

Note that Eqs. (3.33) and (3.34) define an equation for ψ formulated in terms of the Schwarzian derivative.

We find from (3.29) and (3.30), using (3.32) and (3.33), that

$$\begin{aligned} \frac{\varphi_t}{\varphi_x} &= -\frac{\varphi_{xxx}}{\varphi_x} + \frac{3}{4}\frac{\varphi_{xx}^2}{\varphi_x^2} - \frac{3}{2}\frac{\psi_{xxx}}{\psi_x} - \frac{3}{2}\frac{\varphi_{xx}}{\varphi_x}\frac{\psi_{xx}}{\psi_x} \ , \\ \frac{\psi_t}{\psi_x} &= \frac{1}{2}\frac{\psi_{xxx}}{\psi_x} + \frac{3}{2}\frac{\varphi_{xx}}{\varphi_x}\frac{\psi_{xx}}{\psi_x} + \frac{3}{4}\frac{\psi_{xx}^2}{\psi_x^2} \ . \end{aligned} \tag{3.35}$$

These equations may be written in the form

$$\psi_t = A\psi_{xx} + B\psi_x \ , \qquad \psi_{xxx} = -A\psi_{xx} + (C - B)\psi_x \ , \tag{3.36}$$

where

$$A = \varphi_{xx}/\varphi_x \ , \qquad C = -\varphi_t/\varphi_x - \{\varphi; x\} \ , \qquad B = \tfrac{1}{2}A^2 - \tfrac{1}{3}C \ . \tag{3.37}$$

The compatibility condition ($\psi_{txxx} = \psi_{xxxt}$) of the linear equations, (3.36), for ψ obtains the equation for φ,

$$2\frac{\partial}{\partial t}\left(\frac{\varphi_t}{\varphi_x} + \{\varphi; x\}\right) = \frac{\partial}{\partial x}\left(\begin{array}{c} \frac{\partial}{\partial x^2}\left(\frac{\varphi_t}{\varphi_x} + \{\varphi; x\}\right) - \frac{9}{2}\{\varphi; x\}^2 \\ +6\{\varphi; x\}\left(\frac{\varphi_t}{\varphi_x} + \{\varphi; x\}\right) - \left(\frac{\varphi_t}{\varphi_x} + \{\varphi; x\}\right)^2 \end{array}\right) \ , \tag{3.38}$$

which is formulated in terms of the Schwarzian derivative. The equivalent condition ($\varphi_{txxx} = \varphi_{xxxt}$) obtains, from (3.35), that ψ satisfies Eqs. (3.33) and (3.34), which is distinct from (3.38). Therefore (3.35) defines a Bäcklund transformation between two equations formulated in terms of the Schwarzian derivative.

Letting

$$\Omega = \varphi_t/\varphi_x + \{\varphi; x\} \ , \qquad W = \varphi_{xx}/\varphi_x \ , \tag{3.39}$$

the Miura transformation (3.16) is

$$\begin{aligned} -2u_2 &= W_{1x} + \tfrac{1}{2}(W_1^2 + H_1^2) = -\tfrac{3}{2}\Omega \ , \\ -3a\omega_1 &= \tfrac{3}{2}(H_{1x} + W_1 H_1) = \Omega - \tfrac{3}{2}(W_x - \tfrac{1}{2}W^2) \ . \end{aligned} \tag{3.40}$$

Rather than consider Eqs. (3.38) and (3.34) separately, we will define the new variables

$$V = \varphi_{xx}/\varphi_x \ , \qquad W = \psi_{xx}/\psi_x \ , \tag{3.41}$$

and find from (3.35) the equations

$$\begin{aligned} V_t &= \frac{\partial}{\partial x} \circ \left(\frac{\partial}{\partial x} + V\right) \circ \{-V_x + \tfrac{1}{2}V^2 - \tfrac{3}{2}W_x - \tfrac{3}{2}W^2 - \tfrac{3}{2}WV - \tfrac{3}{4}V^2\} \ , \\ W_t &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + W\right) \circ \{\tfrac{1}{2}(W_x - \tfrac{1}{2}W^2) + \tfrac{3}{4}(W+V)^2\} \ . \end{aligned} \tag{3.42}$$

These equations allow the following singularities:

$$V \sim V_0\epsilon^{-1} \ , \qquad W \sim W_0\epsilon^{-1} \tag{3.43}$$

[where for simplicity $\epsilon = x + f(t)$, j is resonance]

$$\begin{array}{llll} \text{(i)} & V_0 = 0 \ , & \omega_0 = 1 \ , & j = -1, 1, 2, 3, 3, 4 \ ; \\ \text{(ii)} & V_0 = -2 \ , & W_0 = -1 \ , & j = -1, 1, 2, 3, 3, 4 \ ; \\ \text{(iii)} & V_0 = 4 \ , & W_0 = -3 \ , & j = -2, -1, 3, 3, 4, 5 \ ; \\ \text{(iv)} & V_0 = -2 \ , & W_0 = 2 \ , & j = -5, -1, 3, 3, 4, 8 \ ; \\ \text{(v)} & V_0 = 2 \ , & W_0 = -2 \ , & j = -1, 1, 2, 3, 3, 4 \ ; \\ \text{(vi)} & V_0 = 2 \ , & W_0 = -3 \ , & j = -2, -1, 3, 3, 4, 5 \ ; \\ \text{(vii)} & V_0 = -2 \ , & W_0 = 2 \ , & j = -1, -1, 3, 3, 4, 4 \ ; \\ \text{(viii)} & V_0 = -6 \ , & W_0 = 2 \ , & j = -5, -1, 3, 3, 4, 8 \ . \end{array} \tag{3.44}$$

The Bäcklund transformations for (3.42) are of the form

$$V = V_0\epsilon^{-1} + V_1 \ , \qquad W = W_0\epsilon^{-1} + W_1 \ . \tag{3.45}$$

For (i), (ii), and (v), with resonances at $j = -1, 1, 2, 3, 3, 4$, (3.45) obtains a system of five equations for the five unknowns $(\epsilon, V_0, W_0, V_1, W_1)$. For (vii) there are six equations in five unknowns. We consider each in turn.

For BT1, we have

$$V = V_1 \ , \qquad W = \epsilon_x/\epsilon + W_1 \ , \tag{3.46}$$

where ϵ satisfies (3.38),

$$\begin{aligned} &\epsilon_x = 1/\varphi_x\psi_x^2 \ , \\ &\frac{\epsilon_t}{\epsilon_x} = \frac{\psi_t}{\psi_x} + \frac{\partial}{\partial x}\left(\frac{\varphi_{xx}}{\varphi_x}\right) - \frac{\varphi_{xx}^2}{\varphi_x^2} - \frac{\psi_{xx}}{\psi_x}\frac{\varphi_{xx}}{\varphi_x} \ , \end{aligned} \tag{3.47}$$

$$V_1 = \varphi_{xx}/\varphi_x \ , \qquad W_1 = \psi_{xx}/\psi_x \ , \tag{3.48}$$

and both (ϵ, ψ) and (φ, ψ) satisfy (3.35). Note (3.47) defines a Bäcklund transformation for Eq. (3.35),

$$(\epsilon, \psi) \leftrightarrow (\varphi, \psi) \ , \tag{3.49}$$

while (3.46) defines the BT

$$(\varphi, \psi) \leftrightarrow (\varphi, \psi') \ , \tag{3.50}$$

where

$$\psi'_x = \epsilon\psi_x \ , \qquad \frac{\psi'_t}{\psi'_x} = \frac{\psi_t}{\psi_x} + \frac{\epsilon_x}{\epsilon}\frac{\varphi_{xx}}{\varphi_x} \ . \tag{3.51}$$

For BT2, we have

$$V = -2(\epsilon_x/\epsilon) + V_1 \ , \qquad W = \epsilon_x/\epsilon + W_1 \ , \tag{3.52}$$

where (ϵ, ψ) satisfy (3.35):

$$V_1 = \epsilon_{xx}/\epsilon_x \ , \qquad W_1 = \psi_{xx}/\psi_x \ . \tag{3.53}$$

Letting

$$V = \varphi'_{xx}/\varphi'_x \ , \qquad W = \psi'_{xx}/\psi'_x \ , \tag{3.54}$$

then

$$\varphi' = -1/\epsilon \ , \tag{3.55}$$

$$\psi'_x = \epsilon\psi_x \ , \qquad \psi'_t = \epsilon\psi_t - \psi_x\epsilon_{xx} - 2\psi_{xx}\epsilon_x \ , \tag{3.56}$$

where (φ', ψ') satisfy (3.35).

For BT5, we have

$$V = 2(\epsilon_x/\epsilon) + V_1 \ , \qquad W = -2(\epsilon_x/\epsilon) + W_1 \ , \tag{3.57}$$

where

$$V_1 = \varphi_{xx}/\varphi_x \ , \qquad W_1 = \epsilon_{xx}/\epsilon_x \tag{3.58}$$

obtains (φ, ϵ) satisfying (3.35) and, with (3.54),

$$\varphi'_x = \epsilon^2\varphi_x \ , \qquad \varphi'_t = \epsilon^2\varphi_t + 4\varphi_x\epsilon\epsilon_{xx} + 2\varphi_{xx}\epsilon\epsilon_x - 2\varphi_x\epsilon_x^2 \ , \tag{3.59}$$

$$\psi' = -1/\epsilon \ . \tag{3.60}$$

Finally, for BT7, we have

$$V = -2(\epsilon_x/\epsilon) + V_1 \ , \qquad W = 2(\epsilon_x/\epsilon) + W_1 \ , \tag{3.61}$$

where

$$V_1 = \epsilon_{xx}/\epsilon_x \ , \qquad W_1 = \psi_{xx}/\psi_x = -\epsilon_{xx}/\epsilon_x \ , \tag{3.62}$$

$$\epsilon_t/\epsilon_x = \tfrac{1}{2}\{\epsilon; x\} \ . \tag{3.63}$$

Note the restriction

$$\psi_x = \epsilon_x^{-1} \ . \tag{3.64}$$

With (3.54),

$$\varphi' = -1/\epsilon \ , \qquad \psi'_x = \epsilon^2\psi_x = \epsilon^2/\epsilon_x \ , \tag{3.65}$$

where (φ', ψ') satisfy (3.63). As expected BT7 is a reduction of (3.35) (to the KdV equation [4]) which is preserved by (3.61).

The Bäcklund transformations, BT1, BT2, BT5, and BT7, form a group under composition that properly restricts to (3.35); that is, maps solutions into solutions. The following identities are easily verified [where $\binom{\varphi'}{\psi'} = \mathrm{BT}\binom{\varphi}{\psi}$]:

$$\begin{aligned}
&\text{(i)} && \mathrm{BT1} \circ \mathrm{BT1} = \mathrm{BT2} \circ \mathrm{BT2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \\
&\text{(ii)} && \mathrm{BT5} \circ \mathrm{BT5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \ , \\
&\text{(iii)} && \mathrm{BT1} \circ \mathrm{BT2} = \mathrm{BT2} \circ \mathrm{BT1} \ , \\
&\text{(iv)} && \mathrm{BT7} \circ \mathrm{BT7} = I \ ,
\end{aligned} \tag{3.66}$$

and subject to the restriction on the domain of BT7,

$$\psi_x = \varphi_x^{-1} \ , \tag{3.67}$$

we also have

(v) $\quad$ $\mathrm{BT1} \circ \mathrm{BT7} = I$. $\quad$ (3.68)

Furthermore, BT5 preserves the KdV restriction, (3.67), while BT1 and BT2 do not [preserve (3.67)]. The Bäcklund transformation

$$K = \mathrm{BT5} \circ \mathrm{BT7} \tag{3.69}$$

generates a sequence of KdV solutions [4]

$$\begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix} = K^j \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} \ , \tag{3.70}$$

where (φ_0, ψ_0), (φ_j, ψ_j) satisfy (3.63) and ($\psi_{0x} = \varphi_{0x}^{-1}$), $\psi_{j,x} = \varphi_{j,x}^{-1}$). For instance,

$$\varphi_0 = \psi_0 = x \tag{3.71}$$

obtains a Sequence of rational KdV solutions [4].

In general, the (φ_j, ψ_j) are distinct in that $K^n \neq I$ for any $n > 0$. Note that

$$K = \mathrm{BT5} \circ \mathrm{BT7} \neq K^t = \mathrm{BT7} \circ \mathrm{BT5} \ . \tag{3.72}$$

Also,

$$\mathrm{BT1} \circ \mathrm{BT5} \neq \mathrm{BT5} \circ \mathrm{BT1} \ , \qquad \mathrm{BT2} \circ \mathrm{BT5} \neq \mathrm{BT5} \circ \mathrm{BT2} \ . \tag{3.73}$$

Application of the group of transformations generated by (BT1, BT2, BT5) to (3.70) produces a "lattice" of solutions of the Hirota-Satsuma equations (3.35). It can be shown that (BT1, BT2, BT5, BT7) map rational solutions into rational solutions [of (3.35)]. Therefore, from (3.70) and (3.71), rational solutions of the Hirota-Satsuma equations are found.

Direct calculation obtains a few solutions:

$$\begin{aligned} \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} &= \begin{pmatrix} x \\ x \end{pmatrix} \overset{\mathrm{BT2}}{\longrightarrow} \begin{pmatrix} -\frac{1}{x} \\ \frac{x^2}{2} \end{pmatrix} \overset{\mathrm{BT5}}{\longrightarrow} \begin{pmatrix} \frac{x^3-24t}{12} \\ -\frac{2}{x^2} \end{pmatrix} \\ &\overset{\mathrm{BT2}}{\longrightarrow} \begin{pmatrix} -\frac{12}{x^3-24t} \\ \frac{x^3+12t}{3x^2} \end{pmatrix} \overset{\mathrm{BT5}}{\longrightarrow} \begin{pmatrix} -\frac{4}{x}\frac{x^3-6t}{x^3-24t} \\ -\frac{3x^2}{x^3+12t} \end{pmatrix} \ , \end{aligned} \tag{3.74}$$

$$\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = \begin{pmatrix}x\\ x\end{pmatrix} \xrightarrow{\text{BT5}} \begin{pmatrix}\frac{x^3-6t}{3}\\ -\frac{1}{x}\end{pmatrix} \xrightarrow{\text{BT2}} \begin{pmatrix}-\frac{3}{x^3-6t}\\ \frac{x^3+12t}{6x}\end{pmatrix}$$
$$\xrightarrow{\text{BT5}} \begin{pmatrix}\frac{x}{4}\frac{x^3-24t}{x^3-6t}\\ \frac{6x}{x^3+12t}\end{pmatrix} \xrightarrow{\text{BT2}} \begin{pmatrix}-\frac{4}{x}\frac{x^3-6t}{x^3-24t}\\ -\frac{3x^2}{x^3+12t}\end{pmatrix}, \tag{3.75}$$

$$\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = \begin{pmatrix}x\\ x\end{pmatrix} \xrightarrow{\text{BT1}} \begin{pmatrix}x\\ \frac{x^2}{2}\end{pmatrix} \xrightarrow{\text{BT5}} \begin{pmatrix}\frac{x^5}{20}\\ -\frac{2}{x^2}\end{pmatrix} \xrightarrow{\text{BT1}} \begin{pmatrix}\frac{x^5}{20}\\ \frac{x^3+12t}{12x^2}\end{pmatrix}$$
$$\xrightarrow{\text{BT5}} \begin{pmatrix}\frac{1}{576}\left(\frac{x^7}{7}+6tx^4+144t^2x\right)\\ -\frac{12x^2}{x^3+12t}\end{pmatrix}, \tag{3.76}$$

$$\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = \begin{pmatrix}x\\ x\end{pmatrix} \xrightarrow{\text{BT5}} \begin{pmatrix}\frac{x^3-6t}{3}\\ -\frac{1}{x}\end{pmatrix} \xrightarrow{\text{BT1}} \begin{pmatrix}\frac{x^3-6t}{3}\\ \frac{x^3+12t}{6x}\end{pmatrix}$$
$$\xrightarrow{\text{BT5}} \begin{pmatrix}\frac{1}{36}\left(\frac{x^7}{7}+6tx^4+144t^2x\right)\\ -\frac{6x}{x^3+12t}\end{pmatrix} \tag{3.77}$$
$$\xrightarrow{\text{BT1}} \begin{pmatrix}\frac{1}{36}\left(\frac{x^7}{7}+6tx^4+144t^2x\right)\\ -\frac{3x^2}{x^3+12t}\end{pmatrix},$$

$$\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = \begin{pmatrix}x\\ x\end{pmatrix} \xrightarrow{\text{BT1}} \begin{pmatrix}x\\ \frac{x^2}{2}\end{pmatrix} \xrightarrow{\text{BT5}} \begin{pmatrix}\frac{x^5}{20}\\ -\frac{2}{x^2}\end{pmatrix}$$
$$\xrightarrow{\text{BT2}} \begin{pmatrix}-\frac{20}{x^5}\\ \frac{x^3+30t}{15}\end{pmatrix} \xrightarrow{\text{BT5}} \begin{pmatrix}\frac{4}{9}\left(\frac{x^6-30tx^3-180t^2}{x^5}\right)\\ -\frac{15}{x^3+30t}\end{pmatrix}, \tag{3.78}$$

From (3.74) and (3.75),

$$(\text{BT5}\circ\text{BT2})^2\circ\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = (\text{BT2}\circ\text{BT5})^2\circ\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix}, \tag{3.79}$$

and, from (3.76) and (3.77),

$$D\circ(\text{BT5}\circ\text{BT1})^2\circ\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix} = (\text{BT1}\circ\text{BT5})^2\circ\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix}, \tag{3.80}$$

where

$$D = \begin{pmatrix}16 & 0\\ 0 & \frac{1}{4}\end{pmatrix}, \tag{3.81}$$

and

$$\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} \ . \tag{3.82}$$

The periodicities (3.79) and (3.80) are not verified in general [for arbitrary $\binom{\psi_0}{\varphi_0}$]. Determination of relationships of the form (3.79) and (3.80) when $\binom{\psi_0}{\varphi_0}$ belongs to the KdV sequence, (3.70), may provide a method for "classifying" the Hirota-Satsuma solutions.

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