Bäcklund Transformation and Linearizations of the Hénon-Heiles System

John Weiss
Center for Studies of Nonlinear Dynamics, La Jolla Institute,
8950 Villa La Jolla Drive Suite 2150, La Jolla, CA 92037, U.S.A.
and Institute for Pure and Applied Physical Science,
University of California, San Diego, La Jolla, CA 92093, U.S.A.

Abstract

When the Hénon-Heiles system possesses the Painlevé property certain Bäcklund transformations are defined in terms of the manifold of singularities. The resulting system of equations are shown to effectively linearize the Hénon-Heiles system.

A system of nonlinear ordinary differential equations is said to possess the Painlevé property when all the "movable" singularities are simple poles [1]. Since the work of Kovalevskaya on the motion of a rigid body about a fixed point, the Painlevé property, in its various guises, has been proposed as a criterion for complete integrability [2]. Recently, it is found that: (1) the Painlevé property is a necessary condition for "algebraically complete integrability" in terms of "abelian functions" [3]. (2) If a system misses the Painlevé property by a certain degree (has complex or irrational "resonances"), the system cannot be algebraically integrable [4].

Now, in [5], we showed that the Painlevé property, when formulated in terms of a "singular manifold", has a natural extension to systems of nonlinear partial differential equations. If the manifold of singularities is determined by

$$\varphi(z_1, z_2, \ldots, z_n) = 0 \tag{1}$$

and $u = u(z_1, \ldots, z_n)$ is a solution of the p.d.e., then it is required that:

$$u = \varphi^{\alpha} \sum_{j=0}^{\infty} u_j \varphi^j \ , \tag{2}$$

where $u_0 \neq 0$, $\varphi = \varphi(z_1, \ldots, z_n)$, $u_j = u_j(z_1, \ldots, z_n)$ are analytic functions of z_j in a neighborhood of (1), and α is a (negative) integer. Also, the manifold (1) is assumed to be "noncharacteristic" so that the "single-valued" expansion (2) about the "movable" singularity (1) will be well defined, in the sense of the Cauchy-Kovalevskaya theorem. When (2) is correct the p.d.e, is said to possess the Painlevé property, and is conjectured to be integrable.

The classical definition of the Painlevé property (for o.d.e.'s) is obtained when

$$\varphi(t) = t - t_0 \ . \tag{3}$$

However, it is still possible (for o.d.e.'s) to allow $\varphi = \varphi(t)$ to be an arbitrary function (i.e., expand about the "zeros" of φ) as long as *** near (1) φ is "noncharacteristic"). Herein, we find that this allows Bäcklund transformation to be defined for ordinary differential equations. In general, Bäcklund transformations are obtained by "truncating" (2) at the "constant" level term. That is, we set:

$$u = u_0 \varphi^{-N} + u_1 \varphi^{-N+1} + \cdots + u_n \ , \tag{4}$$

and find, from the recursion relations for u_j, an overdetermined system of equations for φ' u_j; $j = 0, 1, \ldots, N$, when u_n will satisfy the (original) equation [6, 7, 8, 9]. In [7] Bäcklund transformations were defined, iteratively, for the sequence of higher KdV equations, and it was remarked that this allows the recursive "linearization" of the sequence of "steady state" equations (these being o.d.e.'s). In this letter we find that when the Hénon-Heiles system possesses the Painlevé property the above procedure provides the linearization of this system.

The Hénon-Heiles system

$$\ddot{X} = -AX - 2dXY \ , \qquad \ddot{Y} = -BY + cY^2 - dX^2 \tag{5}$$

is derived from the hamiltonian:

$$H = \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}(AX^2 + BY^2) + dX^2Y - \tfrac{1}{3}cY^3 \ , \tag{6}$$

and is found to possess the Painlevé property when [10, 11]:

case (i): $\quad d/c = -1 \ , \qquad A = B \ ,$ (7)

case (ii): $\quad d/c = -\frac{1}{6} \ ,$ (8)

case (iii): $\quad d/c = -\frac{1}{16} \ , \qquad B = 16A \ .$ (9)

The nontrivial cases are (ii), (iii). (Case (i) is separable.) (See [10] for details.)

In case (ii) ($d/c = -\frac{1}{16}$ Eqs. (5) have an expansion of the form:

$$X = \varphi^{-1} \sum_{j=0}^{\infty} X_j \varphi^j \ , \qquad Y = \varphi^{-2} \sum_{j=0}^{\infty} Y_j \varphi^j \ . \tag{10}$$

The resonances occur at $j = -1, 0, 3, 6$, i.e. (φ, X_0, X_3, Y_6) are the arbitrary constants (functions) of integration. As explained above, to define the B.T., we let

$$X = X_0 \varphi^{-1} + X_1 \ , \qquad Y = Y_0 \varphi^{-2} + Y_1 \varphi^{-1} + Y_2 \ , \tag{ll}$$

where $(\varphi, X_0, X_1, Y_0, Y_1, Y_2)$ are functions of t.

These results, after evaluation

$$\begin{aligned} Y_0 &= -\varphi_t^2 \ , \qquad Y_1 = \varphi_{tt} \ , \\ Y_2 &= \tfrac{1}{4}(4A - B - V - \varphi_{tt}^2/\varphi_t^2) \ , \qquad X_0^2 = \varphi_t^2 V \ , \\ X_1 &= -\tfrac{1}{2}(V_t/V + \varphi_{tt}/\varphi_t) V^{1/2} \ , \end{aligned} \tag{12}$$

where

$$V = \{\varphi; t\} + 3A - \tfrac{1}{2}B \ , \tag{13}$$

$$V_{tt} + \tfrac{3}{2}V^2 = 2A(B - 3A) \ , \tag{14}$$

$$V_t^2 + V^3 = 4A(B - 3A)V \ , \tag{15}$$

and

$$\{\varphi; t\} = (\partial/\partial t)(\varphi_{tt}/\varphi_t) - \tfrac{1}{2}(\varphi_{tt}/\varphi_t)^2 \ .$$

The Bäcklund transformation

$$\begin{aligned} X &= X_0/\varphi + X_1 \ , \\ Y &= (\partial^2/\partial t^2) \ln \varphi + Y_2 \ , \end{aligned} \tag{16}$$

where (X, X_1), (Y, Y_2) satisfy (5), is well defined and the integration of Eqs. (5) is reduced to the system of Eqs. (13) and (15). Equation (15) defines V as a Weierstrass elliptic function depending on one parameter, V_0 (A, B are fixed). Equation (13), which determines φ [and, by (12), (X, X_1), (Y, Y_2)], is expressed in terms of the schwarzian derivative, $\{\varphi; t\}$ [6, 7]. Therefore, if

$$\varphi = U_1/U_2 \tag{17}$$

and (U_1, U_2) are linearly independent solutions of

$$U_{tt} = -\tfrac{1}{2}(V + \tfrac{1}{2}B - 3A)U \ , \tag{18}$$

then Eq. (13) will be satisfied [6]. By the invariance of the schwarzian derivative under the Moebius group:

$$\varphi = (a\psi + b)/(c\psi + d) \ , \tag{19}$$

where $ad - gc = 1$, three parameters are obtained. Thus, the solution of (5) defined in terms of (φ, v) would seem to depend on four parameters and represents the general form of the solution in case (ii). However, since (16) is homogeneous in φ (of degree zero) the transformation, (19), introduces only two arbitrary parameters and the resulting solution will depend, in general, on three parameters. We note that in going from Eq. (14) to Eq. (15) the constant of integration is fixed by the Bäcklund transformation. As was the situation in [7], the Bäcklund transformation "linearizes" (5) by representing the solutions in terms of a Schrödinger equation (18) whose "potential" is defined by an (integrable) equation of lower order, (15).

For case (iii) of the Hénon-Heiles system ($d/c = -\frac{1}{16}$, $B = 16A$) the expansions are:

$$X = \varphi^{-1/2} \sum_{j=0}^{\infty} X_j \varphi^j \ , \qquad Y = \varphi^{-2} \sum_{j=0}^{\infty} Y_j \varphi^j \ , \tag{20}$$

with resonances at $j = -1, 0, 2, 6$. To get single-valued variables we let:

$$Z = X^2 \ , \tag{21}$$

finding

$$\begin{aligned} ZZ_{tt} &= -\tfrac{1}{2} Z_t^2 = -2AZ^2 - 4YZ^2 \ , \\ Y_{tt} &= -16AY - 16Y^2 - Z \ , \end{aligned} \tag{22}$$

where $d = 1$, $c = -16$, $B = 16A$, and

$$Z = \varphi^{-1} \sum_{j=0}^{\infty} Z_j \varphi^j \ , \qquad Y = \varphi^{-2} \sum_{j=0}^{\infty} Y_j \varphi^j \ .$$

The Bäcklund transformation for (Z, Z_1), (Y, Y_2):

$$Z = Z_0 \varphi^{-1} + Z_1 \ , \qquad Y = Y_0 \varphi^{-2} + Y_1 \varphi^{-1} + Y_2 \ , \tag{23}$$

is well defined with:

$$\begin{aligned} &Z_0 = -\tfrac{3}{8} \varphi_t V_t \ , \qquad Z_1 = \tfrac{3}{16} [2V_{tt} + (\varphi_{tt}/\varphi_t) V_t] \ , \\ &Y_0 = -\tfrac{3}{8} \varphi_t^2 \ , \qquad Y_1 = \tfrac{3}{8} \varphi_{tt} \ , \\ &Y_2 = -\tfrac{3}{8} (\tfrac{1}{3} V + \tfrac{1}{4} \varphi_{tt}^2/\varphi_t^2 + \tfrac{4}{3} A) \ , \end{aligned} \tag{24}$$

where

$$V = \{\varphi; t\}\ , \qquad V_{tt} + \tfrac{1}{4}V^2 = 16A^2\ . \tag{25}$$

Again, V is a Weierstrass elliptic function and:

$$\varphi = U_1/U_2\ , \tag{26}$$

where (U_1, U_2) satisfy:

$$U_{tt} = -\tfrac{1}{2}VU\ , \tag{27}$$

defines (φ, V) depending on four arbitrary parameters and, by (24), the general solution of (5) for case (iii).

We note that, for this case, V depends on two parameters [compares Eqs. (25), (14) and (15)].

References

[1] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[2] S. Kovalevskaya, *Acta. Math.* 14 (1890) 81.
V. V. Golubev, *Lectures on the Integration of the Equations of Motion of a Rigid Body*, State Pub. Housde, Moscow, 1953.
M. J. Ablowitz, A. Ramani and H. Segur, *J. Math. Phys.* **19** (1980) 715.
R. Nakach, in *Plasma Physics* (H. Wilhelmsson, ed.), Plenum, New York, 1977.
J. B. McLeod and P. J. Olver, *SIAM J. Math. Anal.* **14** (1983) 488.

[3] M. Adler and P. van Moerbeke, *Invent. Math.* **67** (1982) 297.

[4] H. Yoshida, Necessary condition for the existence of algebraic first integrals. I & II, *Celestial Mech.* (1983), to be published.

[5] J. Weiss, M. Tabor and G. Carnevale, *J. Math. Phys.* **24** (1983) 522. [6]

[6] J. Weiss, *J. Math. Phys.* **24** (1983) 1405.

[7] J. Weiss, On classes of integrable systems and the Painlevé property, *J. Math. Phys.* (1983), to be published.

[8] D. V. Chudnovsky, G. V. Chudnovsky and M. Tabor, *Phys. Lett.* **97A** (1983) 268.

[9] J. Weiss, The sine-Gordon equations: complete and partial integrability, *J. Math. Phys.* (1983), submitted.

[10] Y. F. Chang, M. Tabor and J. Weiss, *J. Math. Phys.* **23** (1982) 531.

[11] T. Bountis, H. Segur and F. Vivaldi, *Phys. Rev.* **A25** (1982) 1257.