The Painlevé Property for Partial Differential Equations.
II. Bäcklund transformation, Lax pairs, and the Schwarzian derivative

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Abstract

In this paper we investigate the Painlevé property for partial differential equations. By application to several well-known partial differential equations (Burgers, KdV, MKdV, Bousinesq, higher-order KdV and KP equations) it is shown that consideration of the "singular manifold" leads to a formulation of these equations in terms of the "Schwarzian derivative". This formulation is invariant under the Moebius group (acting on dependent variables) and is shown to obtain the appropriate Lax pair (linearization) for the underlying nonlinear pde.

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1 Introduction

We say [1] that a partial differential equation has the Painlevé property when the solutions of the pde are "single-valued" about the movable, singularity manifold. To be precise, if the singularity manifold is determined by

$$\phi(z_1, \ldots, z_n) = 0 \tag{1.1}$$

and $u = u(z_1, \ldots, z_n)$ is a solution of the partial differential equation, then we assume that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j \ , \tag{1.2}$$

where

$$\phi = \phi(z_1, \ldots, z_n)\ , \qquad u_j = u_j(z_1, \ldots, z_n)\ , \qquad u_0 \neq 0\ ,$$

are analytic functions of (z_j) in a neighborhood of the manifold (1.1) and α is an integer. Substitution of (1.2) into the partial differential equation determines the value(s) of α and defines the recursion relations for u_j, $j \neq 0, 1, 2, \ldots$. When the ansatz (1.2) is correct, the pde is said to possess the Painlevé property and is conjectured to be integrable.

In this paper we demonstrate a connection between the Painlevé property and the occurrence of the Lax pairs (and Bäcklund transforms) for several well-known nonlinear pde's. These include the Burgers', KdV, MKdV, Bousinesq, higher-order KdV, and Kadomtsev-Petviashvili equations. This connection is most clearly formulated in terms of the "Schwarzian derivative" [2], [3] of the function defining the singular manifold, (1.1). Consideration of the Painlevé property associated with the Schwarzian derivative leads to the identification of a wide class of nonlinear pde's that possess, in part, the Painlevé property. With some care, it appears possible to extend this identification to equations in any (finite) number of independent variables. Study of this class of equations is currently in progress.

In Sec. 2 we develop, at some length, the above procedures for the Korteweg-de Vries equation. Then in the succeeding sections the results for a number of equations are stated.

In Appendix A it is shown how an infinite sequence of "conserved densities", can be defined by the expansion of the solution (of the KdV equation) about the "singularity manifold".

In Appendix B the Harry Dym equation is obtained, by a charge of variables, from the KdV equation.

2 The Korteweg-de Vries Equation

The KdV equation

$$u_t + uu_x + \delta u_{xxx} = 0 \tag{2.1}$$

possesses the Painlevé property[1]. The expansion about the singular manifold has the form

$$u = \phi^{-2} \sum_{j=0}^{\infty} u_j \phi^j\ . \tag{2.2}$$

The recursion relations for the u_j are presented in Appendix A. There it is found that the "resonances" occur at

$$j = -1, 4, 6\ . \tag{2.3}$$

Resonances are those values of j at which it is possible to introduce arbitrary functions into the expansion (2.2) (see [1], [4]). For each nontrivial resonance there occurs a compatibility condition that must be satisfied if the solution is to have a single-valued expansion. The resonance at $j=-1$ corresponds to the "arbitrary" function ϕ defining the singular manifold (2.1). The compatibility conditions at $j=4$ and 6 are satisfied identically [1].

From the recursion relations, we find:

$$j=0: \qquad u_0 = -12\delta\phi_x^2 \ ; \tag{2.4}$$

$$j=1: \qquad u_1 = 12\delta\phi_{xx} \ ; \tag{2.5}$$

$$j=2: \qquad \phi_x\phi_t + \phi_x^2 u_2 + 4\delta\phi_x\phi_{xxx} - 3\delta\phi_{xx}^2 = 0 \ ; \tag{2.6}$$

$$j=3: \qquad \phi_{xt} + \phi_{xx}u_2 - \phi_x^2 u_3 + \delta\phi_{xxxx} = 0 \ ; \tag{2.7}$$

$j=4:$ compatibility condition:

$$\frac{\partial}{\partial x}(\phi_{xt} + \delta\phi_{xxxx} + \phi_{xx}u_2 - \phi_x^2 u_3) = 0 \ . \tag{2.8}$$

By (2.7) the compatibility condition (2.8) is satisfied identically. The compatibility condition at $j=6$ is also satisfied identically [1], and the KdV equation is thus shown to be Painlevé.

We now specialize (2.2) by setting the "resonance" functions

$$u_4 = u_6 = 0 \ . \tag{2.9}$$

Furthermore, by requiring

$$u_3 = 0 \ , \tag{2.10}$$

it is easily demonstrated that

$$u_j = 0 \ , \qquad j \geq 3 \ , \tag{2.11}$$

if

$$u_{2t} + u_2 u_{2x} + \delta u_{2xxx} = 0 \ . \tag{2.12}$$

We thus obtain the following Bäcklund transformation [5]:

$$u = 12\delta\frac{\partial^2}{\partial x^2}\ln\phi + u_2 \ , \tag{2.13}$$

where u and u_2 satisfy (2.1) and

$$\phi_x\phi_t + \phi_x^2 u_2 + 4\delta\phi_x\phi_{xxx} - 3\delta\phi_{xx}^2 = 0 \ , \tag{2.14}$$

$$\phi_{xt} + \phi_{xx} + \delta\phi_{xxxx} = 0 \ . \tag{2.15}$$

In [1] it was shown that the substitution

$$\phi_x = v^2 \tag{2.16}$$

reduces (2.14) and (2.15) to the Lax pair (in v) for the KdV equation. However, here we shall proceed in a somewhat different manner. By eliminating u_2 from (2.14) and (2.15) we find that

$$\frac{\partial}{\partial x}\left[\frac{\phi_t}{\phi_x} + \delta\left\{+\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right\}\right] = 0 \tag{2.17}$$

or

$$\frac{\phi_t}{\phi_x} + \delta\{\phi; x\} = \lambda \ , \tag{2.18}$$

where

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$$

is the "Schwarzian derivative" of ϕ [2]. Equation (2.18) is invariant under the Moebius group [3]. That is, if

$$\psi = (a\phi + b)/(c\phi + d) \ , \qquad ad - bc \neq 0 \ , \tag{2.19}$$

and ϕ satisfies (2.18), then ψ satisfies (2.18). This invariance and other well-known properties of the Schwarzian derivative suggests that the substitution

$$\phi = v_1/v_2 \ , \tag{2.20}$$

where (v_1, v_2) satisfy the same system of linear equations, might be of use. With the notation

$$\theta_x = v_2 v_{1x} - v_1 v_{2x} \ , \tag{2.21}$$

$$\psi_t = v_2 v_{1t} - v_1 v_{2t} \ , \tag{2.22}$$

it is found that

$$\frac{\psi_t}{\theta_x} + \delta\left(\{\theta; x\} - 2\frac{v_{2xx}}{v_2} + 2\frac{\theta_{xx}}{\theta_x}\frac{v_{2x}}{v_2}\right) = \lambda \ . \tag{2.23}$$

If (v_1, v_2) satisfy

$$v_{xx} = av \ , \tag{2.24}$$

$$v_t = bv_x + cv \ , \tag{2.25}$$

then

$$\theta_{xx} = 0 \tag{2.26}$$

and

$$\psi_t = b\theta_x \ . \tag{2.27}$$

We find, from (2.23), that

$$b = 2\sigma a + \lambda \ . \tag{2.28}$$

By requiring

$$v_{xxt} = v_{txx} \tag{2.29}$$

and using (2.28), it is found that (2.24) and (2.25) are just the Lax pair for the KdV equation.

We note that the Bäcklund transform defined by (2.13) is essentially the formula employed in [6] for adding or subtracting a bound state to the potential function of the scattering problem (2.24). Furthermore, the constant term λ that appears in Eq. (2.18) is equivalent to the spectral parameter in the second-order scattering problem (2.24). Since the KdV equation is invariant under the Galilean transformation

$$u \to u + \lambda \ , \qquad x \to x + \lambda t \ , \qquad t \to t \ , \tag{2.30}$$

we find that if

$$\phi_t/\phi_x + \delta\{\phi; x\} = 0 \ , \tag{2.31}$$

then

$$\psi(x,t) = \phi(x + \lambda t, t) \tag{2.32}$$

will satisfy (2.18). This observation will be employed in the next section.

3 The Modified KdV Equation

The modified KdV equation

$$w_t + \delta \frac{\partial}{\partial x}\left(w_{xx} - \frac{w^3}{2} \right) = 0 \tag{3.1}$$

possesses the Painlevé property [1].

In general,

$$w = \phi^{-1} \sum_{j=0}^{\infty} w_j \phi^j \tag{3.2}$$

with resonances at

$$j = -1, 3, \text{ and } 4 \ . \tag{3.3}$$

Since

$$w_2 = w_3 = w_4 = 0 \tag{3.4}$$

and w_1 satisfying (3.1) implies

$$w_j = 0 \ , \qquad j \geq 2 \ , \tag{3.5}$$

the associated Bäcklund transform takes the following form:

$$w = 2\phi_x/\phi + w_1 \ , \tag{3.6}$$

where

$$w_1 = \phi_{xx}/\phi_x \tag{3.7}$$

and (w, w_1) satisfy (3.1).

By substitution of (3.7) into (3.1) it is found that

$$\phi_t/\phi_x + \delta\{\phi; x\} = 0 \ , \tag{3.8}$$

where $\{\phi; x\}$ is the Schwarzian derivative. Now this [Eq. (3.8)] is just Eq. (2.18) with $\lambda = 0$. Formally, the Bäcklund transform associated with the modified KdV equation [(3.6) and (3.7)] is degenerate since it does not depend on an arbitrary parameter (the spectral parameter vanishes). Iteration of (3.6) once recreates the initial solution. However, as was noted in the previous section, solutions of (3.8) are simply related to solutions of (2.18) by a partial Galilean transform:

$$x \to x + \lambda t \ . \tag{3.9}$$

Under the transformation (3.9), the MKdV becomes

$$w_t + \delta \frac{\partial}{\partial x}\left(w_{xx} - \frac{w^3}{2}\right) = \lambda w_x \ , \tag{3.10}$$

and the Bäcklund transform reads

$$w = 2\phi_x/\phi + w_1 \tag{3.11}$$

where

$$w_1 = \phi_{xx}/\phi_x \tag{3.12}$$

and

$$\phi_t/\phi_x + \delta\{\phi; x\} = \lambda \ . \tag{3.13}$$

Now identifying Eqs. (3.13) and (2.18), we find that

$$\text{MKdV:} \quad w - w_1 = \tfrac{2\phi_x}{\phi} \ , \tag{3.14}$$

$$\text{KdV:} \quad u - u_2 = 12\delta \tfrac{\partial}{\partial x} \left(\tfrac{\phi_x}{\phi} \right) \ , \tag{3.15}$$

and

$$u - u_2 = 6\delta \frac{\partial}{\partial x} (w - w_1) \ . \tag{3.16}$$

In addition, using (2.6), we find

$$u_2 + \lambda = 3\delta(w_{1,x} - \tfrac{1}{2} w_1^2) \ , \tag{3.17}$$

$$u + \lambda = 3\delta(w_x - \tfrac{1}{2} w^2) \ . \tag{3.18}$$

(3.17) and (3.18) are Miura transformations, mapping solutions of the MKdV into solutions of the KdV equation. Equations (3.14)–(3.18) indicate that the Miura transformation "commutes" with the Bäcklund transforms defined above. We note that Eq. (3.16) may be used to invert the Miura transformation.

To complete the Bäcklund transform for the MKdV, we apply the reverse, partial Galilean transform [after application of (3.9) and (3.11)]

$$x \rightarrow x - \lambda t \tag{3.19}$$

to arrive at the original equation (3.1). This completes the Bäcklund transform for the MKdV.

4 The Bousinesq Equation

The Bousinesq equation

$$u_{tt} + \frac{\partial^2}{\partial x^2} \left(\frac{u_{xx}}{3} + u^2 \right) = 0 \tag{4.1}$$

possesses the Painlevé property [1].

In general, it is found that

$$u = \phi^{-2} \sum_{j=0}^{\infty} u_j \phi^j \ , \tag{4.2}$$

where the resonances occur at

$$j = -1, 4, 5, 6 \ . \tag{4.3}$$

The compatability conditions at $j = 4, 5$, and 6 are satisfied identically. We will require that

$$u_3 = 0 \tag{4.4}$$

and set

$$u_4 = u_5 = u_6 = 0 \tag{4.5}$$

(implying $u_j = 0 \geq j \geq 3$). We find the associated Bäcklund transform

$$u = 2\frac{\partial^2}{\partial x^2} \ln \phi + u_2 \ , \tag{4.6}$$

where (u, u_2) satisfy (4.1), and

$$\phi_t^2 - \phi_{xx}^2 + \tfrac{4}{3}\phi_x \phi_{xxx} + 2u_2 \phi_x^2 = 0 \ , \tag{4.7}$$

$$\phi_{tt} + \tfrac{1}{3}\phi_{xxxx} + 2\phi_{xx} u_2 = 0 \ . \tag{4.8}$$

Now, by eliminating u_2 in (4.7) and (4.8), it is found that

$$\frac{\partial}{\partial t}\left(\frac{\phi_t}{\phi_x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\phi_t}{\phi_x}\right)^2 + \frac{1}{3}\frac{\partial}{\partial x}\{\phi; x\} = 0 \ , \tag{4.9}$$

where $\{\phi; x\}$ is the Schwarzian derivative.

As before, this equation is invariant under the Moebius group. We let

$$\phi = v_1 / v_2 \tag{4.10}$$

and find

$$\frac{\partial}{\partial t}\left(\frac{\psi_t}{\theta_x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\psi_t}{\theta_x}\right)^2 + \frac{1}{3}\frac{\partial}{\partial x}\left(\{\theta; x\} - 2\frac{v_{2xx}}{v_2} + 2\frac{\theta_{xx}}{\theta_x}\frac{v_{2x}}{v_2}\right) = 0 \ , \tag{4.11}$$

where

$$\theta_x = v_2 v_{1x} - v_1 v_{2x} \ , \tag{4.12}$$

$$\psi_t = v_2 v_{1t} - v_1 v_{2t} \ , \tag{4.13}$$

and

$$\{\theta; x\} = \frac{\partial}{\partial x}\left(\frac{\theta_{xx}}{\theta_x}\right) - \frac{1}{2}\left(\frac{\theta_{xx}}{\theta_x}\right)^2 \ . \tag{4.14}$$

To solve (4.11), first we let (v_1, v_2) satisfy the following:

Second-order scattering problem:

$$v_{xx} = av \ , \tag{4.15}$$

$$v_t = bv_x + cv \ . \tag{4.16}$$

Then,

$$\theta_{xx} = 0 \ ,$$

$$\psi_t = b\theta_x \ ,$$

and from (4.11)

$$b_t + bb_x = \tfrac{2}{3}ax \ , \tag{4.17}$$

while by compatability of (4.15) and (4.16)

$$\begin{aligned} &2c_x + b_{xx} = 0 \ , \\ &a_t = c_{xx} + 2ab_x + ba_x \ , \end{aligned} \tag{4.18}$$

or, collecting equations,

$$\begin{aligned} a_t &= -\tfrac{1}{2}b_{xxx} + 2ab_x + ba_x \ , \\ b_t &= -bb_x + \tfrac{2}{3}a_x \ . \end{aligned} \tag{4.19}$$

Unfortunately, it is not possible to introduce an arbitrary,spectral parameter

$$a = u + \lambda \tag{4.20}$$

into Eq. (4.15) without obtaining from (4.19) equations that will depend explicitly on that parameter. Therefore, we shall consider the following:

Third-order scattering problem:

$$v_{xxx} = av_x + bv \ , \tag{4.21}$$

$$v_t = cv_{xx} + dv_x + ev \ . \tag{4.22}$$

Then,

$$\psi_t = c\theta_{xx} + d\theta_x \ , \tag{4.23}$$

and substitution into (4.11) determines

$$c^2 = 1 \ , \qquad d = 0 \tag{4.24}$$

$$ce_x + \tfrac{2}{3}a_x = 0 \ . \tag{4.25}$$

By the consistency condition for (4.21) and (4.22) we find

$$\begin{aligned} &3e_x + 2ca_x = 0 \ , \\ &a_t = 3e_{xx} + 3cb_x + ca_{xx} \ , \\ &b_t + ae_x = e_{xxx} + cb_{xx} \ . \end{aligned} \tag{4.26}$$

Letting

$$a = -\tfrac{3}{2}ce \ , \tag{4.27}$$

we find

$$ce_{tt} + \frac{\partial^2}{\partial x^2}\left(e^2 + c\frac{e^{xx}}{3}\right) = 0 \ , \tag{4.28}$$

where

$$c^2 = 1 \ .$$

Let

$$c = 1 \ ;$$

then

$$\begin{aligned} &e \text{ is a solution of the Bousinesq equation} \\ &a = -\tfrac{3}{2}e \ , \\ &b = \lambda - \tfrac{3}{4}e_x^2 - \tfrac{3}{4}\int^x e_t \ , \\ &c = 1 \ , \qquad d = 0 \ . \end{aligned} \tag{4.29}$$

Equations (4.29) determine the Lax pair for the Bousinesq equation [7]. We note the appearance of the spectral parameter λ.

5 A Higher-Order Korteweg-de Vries Equation

The higher-order KdV equation [8]

$$u_t = \frac{1}{4}\frac{\partial}{\partial_x}(u_{xxxx} + 5u_x^2 + 10u_{xx}u + 10u^3) \tag{5.1}$$

has an expansion, about the singular manifold, of the form

$$u = \phi^{-2}\sum_{j=0}^{\infty} u_j\phi^j \ . \tag{5.2}$$

The resonances occur at

(i) $\quad r=-1,2,5,6,8, \qquad \text{when } u_0=-2\phi_x^2 \qquad (5.3)$

(ii) $\quad r=-3,-1,6,8,10, \quad \text{when } u_0=-6\phi_x^2\ . \qquad (5.4)$

We note the appearance of two solution branches, with branch (i) depending on five, and branch (ii) depending on four, arbitrary functions [1], [3]. In this section we only consider branch (i).

Rather than verify by tedious calculation that the equation possesses the Painlevé property, we shall assume that the equation has an associated Bäcklund transform

$$u=-2\phi_x^2/\phi^2+u_1/\phi+u_2\ . \tag{5.5}$$

By substitution in (5.1) we find that

$$u_0=-2\phi_x^2\ , \qquad u_1=2\phi_{xx}\ , \tag{5.6}$$

$$u_2=-\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right)+\frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2\right]+\lambda\ , \tag{5.7}$$

$$\begin{aligned}8\frac{\phi_t}{\phi_x}= &\ 12\frac{\phi_{xxxxx}}{\phi_x}-10\frac{\phi_{xx}\phi_{xxxx}}{\phi_x^2}+60\frac{\phi_{xx}}{\phi_x}u_{2x}+60u_2^2+20u_{2xx}\\ &+20\left(\frac{4\phi_{xxx}}{\phi_x}-\frac{3\phi_{xx}^2}{\phi_x^2}\right)u_2\ ,\end{aligned} \tag{5.8}$$

and u_2 must satisfy (5.1).

To verify the consistency of the above Bäcklund transform, we substitute (5.7) and (5.8) and obtain

$$4\frac{\phi_t}{\phi_x}=\frac{\partial^2}{\partial x^2}\{\phi;x\}+\tfrac{3}{2}\{\phi;x\}^2+10\lambda\{\phi;x\}+30\lambda^2\ , \tag{5.9}$$

where

$$\{\phi;x\}=\frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right)-\frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$$

is the Schwarzian derivative.

From Eq. (5.9) we readily obtain the Lax pair. Letting

$$\phi=v_1/v_2\ , \tag{5.10}$$

and

$$v_{xx}=av\ , \qquad v_t=bv_x+cv\ , \tag{5.11}$$

we find from (5.9) that

$$4b = -2a_{xx} + 6a^2 - 20\lambda a + 30\lambda^2 \ . \tag{5.12}$$

With the consistency conditions

$$\begin{aligned} &2c_x + b_{xx} = 0 \ , \\ &a_t = c_{xx} + 2ab_x + ba_x \ , \\ &a = \lambda - u \ , \end{aligned} \tag{5.13}$$

we find

$$\begin{aligned} b &= \tfrac{1}{2}u_{xx} + \tfrac{3}{2}u^2 + 2\lambda u + 4\lambda^2 \ , \\ c &= -\tfrac{1}{2}b_x + \alpha \ . \end{aligned} \tag{5.14}$$

This defines the Lax pair for (5.1) [8].

6 The Kadomtsev-Petviashvili, or Two-Dimensional KdV, Equation

The KP equation [9]

$$u_{tx} + u_x^2 + uu_{xx} + \sigma u_{xxxx} + u_{yy} = 0 \tag{6.1}$$

possesses the Painlevé property [1].

It is found [1] that

$$u = \phi^{-2} \sum_{j=0}^{\infty} u_j \phi^j \ , \tag{6.2}$$

with resonances at

$$j = -1, 4, 5, 6 \ . \tag{6.3}$$

The compatability conditions at $j = 4, 5$, and 6 are satisfied identically. We set (truncate at constant level)

$$u_3 = u_4 = u_5 = u_6 = 0 \ ,$$

The associated Bäcklund transform is

$$u = 12\sigma \frac{\partial^2}{\partial x^2} \ln \phi_u^2 \ , \tag{6.4}$$

where

$$\phi_t \phi_x + 4\sigma \phi_x \phi_{xxx} - 3\sigma \phi_{xx}^2 + \phi_y^2 + u_2 \phi_x^2 = 0 \tag{6.5}$$

and

$$\phi_{xt} + \sigma\phi_{xxxx} + \phi_{yy} + \phi_{xx}u_2 = 0 \ . \tag{6.6}$$

Solving for ϕ, we find that

$$\frac{\partial}{\partial x}\left(\frac{\phi_t}{\phi_x} + \sigma\{\phi;x\}\right) + \frac{\partial}{\partial y}\left(\frac{\phi_y}{\phi_x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\phi_y^2}{\phi_x^2}\right) = 0 \ . \tag{6.7}$$

We note that Eq. (6.7) is a combination of Eq. (2.17) for the KdV equation and Eq. (4.9) for the Bousinesq equation. The Lax pair [9] can be readily obtained from (6.7).

7 Burgers' Equation

Burgers' equation

$$u_t + uu_x = \sigma u_{xx} \tag{7.1}$$

possesses the Painlevé property[1]. In general

$$u = \phi^{-1}\sum_{j=0}^{\infty} u_j\phi^j \tag{7.2}$$

with resonances at

$$j = -1, 2 \ . \tag{7.3}$$

The associated Bäcklund transform is

$$u = -2\sigma\phi_x/\phi + u_1 \ , \tag{7.4}$$

where (u, u_1) satisfy (7.1) and

$$\phi_t + u_1\phi_x = \sigma\phi_{xx} \ . \tag{7.5}$$

From (7.1) and (7.5) we find

$$\frac{\partial}{\partial t}\left(\frac{\phi_t}{\phi_x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\phi_t}{\phi_x}\right)^2 - 2\sigma\frac{\partial^2}{\partial x^2}\left(\frac{\phi_t}{\phi_x}\right) + \sigma^2\frac{\partial}{\partial x}\{\phi;x\} = 0 \ . \tag{7.6}$$

where $\{\phi;x\}$ is the Schwarzian derivative.

As before, we let

$$\phi = v_1/v_2 \tag{7.7}$$

and find

$$\frac{\partial}{\partial t}\left(\frac{\psi_t}{\theta_x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\psi_t}{\theta_x}\right)^2 - 2\sigma\frac{\partial^2}{\partial x^2}\frac{\psi_t}{\theta_x} + \sigma^2\frac{\partial}{\partial x}\left(\{\phi; x\} - 2\frac{v_{2xx}}{v_2} + 2\frac{\theta_{xx}}{\theta_x}\frac{v_{2x}}{v_2}\right) = 0 \ . \tag{7.8}$$

If (v_1, v_2) satisfy

$$v_{xx} = av \ , \tag{7.9}$$

$$v_t = bv_x + cv \ , \tag{7.10}$$

then (7.8) obtains

$$b_t - bb_x - 2\sigma b_{xx} = 2\sigma^2 a_x \ , \tag{7.11}$$

while, by cross-differentiation,

$$\begin{aligned} &2c_x + b_{xx} = 0 \ , \\ &a_t = c_{xx} + 2ab_x + ba_x \ , \end{aligned} \tag{7.12}$$

or

$$\begin{aligned} &b_t - bb_x - 2\sigma b_{xx} = 2\sigma^2 a_x \ , \\ &a_t = -\tfrac{1}{2}b_{xxx} + 2ab_x + ba_x \ . \end{aligned} \tag{7.13}$$

If b is functionally independent of a, it is, in general, not possible to introduce as spectral parameter into the above without obtaining equations that depend on this parameter. However, if we set

$$a = -(1/\sqrt{2}\,)b_x + \tfrac{1}{2}b^2 \tag{7.14}$$

and

$$\sigma = 1/\sqrt{2} \ ,$$

then the system (7.13) becomes

$$b_t = 2bb_x + (1/\sqrt{2}\,)b_{xx} \ . \tag{7.15}$$

The Lax pair for this equation is

$$\begin{aligned} &v_{xx} = [-(1/\sqrt{2}\,)b_x + \tfrac{1}{2}b^2]v \ , \\ &v_t = bv_x - \tfrac{1}{2}b_x v \ . \end{aligned} \tag{7.16}$$

By the consistency condition of Eqs. (7.16),

$$[-(1/\sqrt{2}\,)\partial_x + b][b_t - 2bb_x - (1/\sqrt{2}\,)b_{xx}] = 0 \ . \tag{7.17}$$

Letting

$$b = u + \lambda\ , \qquad x \to -2\lambda t\ ,$$

we find

$$\begin{aligned} u_t &= 2uu_x + (1/\sqrt{2})u_{xx}\ , \\ v_{xx} &= [-(1/\sqrt{2})u_x + \tfrac{1}{2}u^2 + u\lambda + \tfrac{1}{2}\lambda^2]v\ , \\ v_t &= (u-\lambda)v_x - \tfrac{1}{2}u_x v\ , \end{aligned} \tag{7.18}$$

where λ is the spectral parameter.

8 The Schwarzian Derivative

The Schwarzian derivative

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{8.1}$$

is a third-order, differential expression which is invariant in form under the Moebius group acting on the dependent variable. That is,

$$\left\{\frac{a\phi+b}{c\phi+d}; x\right\} \equiv \{\phi; x\}\ . \tag{8.2}$$

Furthermore,

$$\{\phi; x\} = \{\psi; x\} + \{f; \psi\}\psi_x^2\ , \tag{8.3}$$

$$\{\phi; x\} = h'^2\{\phi; z\} + \{h; x\}\ , \tag{8.4}$$

where

(i) $\phi = f(\psi)$ is an arbitrary change of dependent variable,

(ii) $z = h(x)$ is an arbitrary change of independent variable.

The Schwarzian derivative appears in a variety of contexts, including conformal mapping of curvilinear polygons [10], [11], algebraic solutions in differential equations [10], [12], and the theory of automorphic functions [11].

In the preceding sections we have considered the Painlevé property and Bäcklund transformations for a number of integrable partial differential equations. Specifically, it is found that the equation for the "singular surface" (in the presence of a Bäcklund transform) is most naturally expressed in terms of the Schwarzian derivative. These equations are invariant under the Moebius group. By subjecting them to the well-known [2] "linearizing" transformation

$$\phi = v_1/v_2\ , \tag{8.5}$$

the Lax pairs for these equations are readily found.

In this section we investigate the Painlevé property associated with equations formulated in terms of the Schwarzian derivative. By this means we seek to identify classes of integrable partial differential equations.

To begin, consider the equation associated with the KdV–MKdV equations, (Secs. 2 and 3).

$$\phi_t/\phi_x + \sigma\{\phi; x\} = \lambda \ , \tag{8.6}$$

where

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\frac{\phi_{xx}^2}{\phi_x^2} \ .$$

This equation is homogeneous in ϕ, and consists of two terms; one, ϕ_t/ϕ_x invariant under arbitrary changes of dependent variable, $\phi = f(\psi)$; the other, a function of the Schwarzian derivative, invariant under the Moebius group,

$$\phi = (a\psi + b)/(c\psi + d) \ . \tag{8.7}$$

In consequence, (8.6) is invariant under (8.7). It is to be noted that all the equations [(2.18), (3.8), (4.9), (5.9), (6.6), and (7.6)] that were found have this structure.

Following the procedure defined before, we find for (8.6) that

$$\phi = \psi^{-1}\sum_{j=0}^{\infty}\phi_j\psi^j \tag{8.8}$$

with resonances at

$$j = -1, 0, 1 \ . \tag{8.9}$$

The compatability conditions at $j = 0$ and $j = 1$ are satisfied and Eq. (8.6) has the Painlevé property about the "movable" singularity (8.8).

The Bäcklund transform for (8.6) is

$$\phi = \pi_0/\psi + \phi_1 \ . \tag{8.10}$$

Direct calculation reveals that the most general form allowed for (8.10) is the transformation (8.7), the Moebius group.

Furthermore, every equation for the singular surface considered in this paper has an expansion of the form (8.8) and possesses the Painlevé property about singularities of this type. We note that the vanishing of ϕ_x, in (8.1) introduces the possibility of movable essential singularities in these equations [e.g., (8.6)] and "manifolds of indeterminacy" [3] in the corresponding a

Bäcklund transforms [i.e., (2.13)]. Therefore, equations of this type cannot be said to possess the Painlevé property identically.

Nevertheless, it is possible to identify a wide class of equations whose only singularities of finite degree are of the form (8.8) and which possess the Painlevé property about these singularities

. It is fairly easy to show that an equation of the form

$$A\left(\frac{\phi_t}{\phi_x}\right) + B(\{\phi;x\}) = 0 \ , \tag{8.11}$$

where $A(\phi_t/\phi_x)$ and $B(\{\phi;x\})$ are constant coefficient multinomials in $(\partial^k/\partial t^j \partial x^i)(\phi_t/\phi_x)$ and $(\partial^l/\partial t^m \partial x^n)\{\phi;x\}$, respectively, with $i+j=k$, $m+n=l$, $\max(k) < \max(l)+2$, will have these properties.

Indeed, any equation of the form

$$A(\phi_{x_1}, \phi_{x_2}, \ldots, \phi_{x_n}) + B(\{\phi;x_1\}, \{\phi;x_2\}, \ldots, \{\phi;x_n\}) = 0 \ , \tag{8.12}$$

where (i) A is a constant coefficient multinomial in

$$\frac{\partial^N}{\partial^{j_1}_{x_1} \cdots \partial^{j_n}_{x_n}} \left(\frac{\phi_{x_1}}{\phi_{x_m}}\right) \ , \qquad j_1 + \cdots + j_n = N \ ,$$

and (ii) B is a constant coefficient multinomial in

$$\begin{aligned} &\frac{\partial^m}{\partial x_1^{j_1} \cdots \partial^{j_n}_{x_n}} \{\phi; x_k\} \ , \\ &j_1 + \cdots + j_m = m \ , \qquad \max(N) < \max(m) + 2 \ , \end{aligned} \tag{8.13}$$

will have the Painlevé property about the movable poles (or order— 1). (Note that we require the highest-order derivatives occur in expressions involving the Schwarzian derivative.)

We conclude this section with an example of an equation of the form (8.10) that is related to nonintegrable equations:

$$\frac{\phi_t}{\phi_x} + \frac{\partial}{\partial x}\{\phi;x\} = 0 \ . \tag{8.14}$$

About the "movable poles" this equation has a solution of the form:

$$\phi = \psi^{-1} \sum_{j=0}^{\infty} \phi_j \psi^j \ , \tag{8.15}$$

with resonances at

$$j = -1, 0, 1, 2 \ . \tag{8.16}$$

It can be readily verified that the compatability conditions are satisfied. (The solution has the Painlevé property about the movable poles.) On the other hand, if we attempt to "integrate" (8.14) by

$$\phi = v_1/v_2 \ , \qquad v_{xx} = av \ , \tag{8.17}$$

and

$$v_t = bv_x + cv \ , \tag{8.18}$$

we find

$$v_{xx} = \tfrac{1}{2}wv \ ; \qquad v_t = w_x v_x - \tfrac{1}{2}w_{xx}v \ , \tag{8.19}$$

and

$$w_t + w_{xxxx} = 2ww_{xx} + w_x^2 \ . \tag{8.20}$$

Unfortunately, it does not appear to be possible to introduce a spectral parameter into (8.19) and (8.20) in such a manner that Eq. (8.19) will hot depend on this parameter. This restricts the type of solution that can be represented by (8.17) to be of a special type.

About the singularities of Eq. (8.20) we find that

$$w \simeq w_0/\psi^2 + \cdots \tag{8.21}$$

with resonances at

$$r = -1 \ , \qquad (7 \pm i\sqrt{11}\,)/2, 8 \ . \tag{8.22}$$

There exists a "Painlevé type" of expansion

$$w = \sum_{j=0}^{\infty} w_j \psi^{j-1} \ , \tag{8.23}$$

where the compatability condition at $j = 8$ is satisfied identically; (8.23) represents a "special" form of the solution since the arbitrary functions introduced at the resonances $r = (7 \pm \sqrt{11}\,)/2$ are not included in (8.23).

Furthermore, by the substitution

$$V = \phi_{xx}/\phi_x \ , \tag{8.24}$$

we find from (8.14) that

$$V_t + \frac{\partial}{\partial x}(V_{xxx} - V_x^2 - V^2 V_x) = 0 \ . \tag{8.25}$$

For this equation we find, by a leading order analysis,

$$V \sim V_0 \phi^{-1} \ , \tag{8.26}$$

where

$$V_0 = 3\phi_x, \ -2\phi_x \ .$$

Letting

$$V_0 = 3\phi_x \ , \tag{8.27}$$

we find the resonances

$$r = -1, \ (7 \pm i\sqrt{11}\,)/2, \ 4 \ . \tag{8.28}$$

Again, the expansion is Painlevé at $r = 4$ and the equation allows the "special" Painlevé form of solution

$$V = \sum_{j=j0}^{\infty} V_j \phi^{j-1} \ . \tag{8.29}$$

We note that the complex resonances in (8.22) and (8.28) are identical.

We now let

$$V_0 = -2\phi_x \tag{8.30}$$

and find the resonances

$$r = -1, 2, 4, 5 \ . \tag{8.31}$$

In this case, we find that the equation has the Painlevé expansion

$$V = \sum_{j=0}^{\infty} V_j \phi^{j-1} \ , \tag{8.32}$$

where (ϕ, V_2, V_4, V_5) are arbitrary.

By forming the Bäcklund transform

$$V = V_0/\phi + V_1 \tag{8.33}$$

from (8.32), we find that

$$\begin{aligned} V_0 &= -2\phi_x \ , \\ V_1 &= \phi_{xx}/\phi_x \ , \end{aligned} \tag{8.34}$$

(V, V_1) satisfy (8.25) and

$$\frac{\phi_t}{\phi_x} + \frac{\partial}{\partial x}\{\phi; x\} = 0 \ . \tag{8.35}$$

Equation (8.35) is just Eq. (8.14).

If one attempts to form a Bäcklund transform (8.33) for the expansion (8.29), there is found

$$V_0 = 3\phi_x \ , \qquad V_1 = -\tfrac{3}{2}(\phi_{xx}/\phi_x) \ . \tag{8.36}$$

(V, V_1) satisfy (8.25) and

$$\phi_t = 0 \ , \qquad \{\phi; x\} = 0 \ . \tag{8.37}$$

Although Eq. (8.14) does not lead to "completely integrable" equations, it is somewhat curious that it does determine equations with somewhat "special" properties.

We feel that further study of equations of the form (8.10) is warranted.

Appendix A: Recursion Relations for the KdV Equation and Conserved Densities

For the KdV equation

$$u_t + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + u_{xx}\right) = 0 \tag{A1}$$

we get

$$u = \phi^{-2}\sum_{j=0}^{\infty} u_j \phi^j \tag{A2}$$

and find

$$u_{j-3,t} + (j-4)\phi_t u_{j-2} + \psi_{j-1,x} + (j-4)\phi_x \psi_j = 0 \ , \tag{A3}$$

where

$$\psi_j = \sum_{m=0}^{j} \frac{u_{j-m}u_m}{2} + \theta_{j-1,x} + (j-3)\phi_x\theta_j \tag{A4}$$

and

$$\theta_j = u_{j-1,x} + (j-2)\phi_x u_j \ . \tag{A5}$$

In (A2), (ϕ, u_4, u_6) are arbitrary.

In this appendix we show how the recursion relations (A3) may be used to formally define an infinite set of conserved densities for the KdV equation. By repeated application of (A3) for $j, j+1, \ldots$ it is easily demonstrated that

$$\frac{\partial}{\partial t}A_j + \frac{\partial}{\partial x}B_j = 0 \ , \tag{A6}$$

where

$$A_j = u_{j-3} + (j-4)\phi u_{j-2} + (j-4)(j-3)(\phi^2/2!)u_{j-1} + (j-4)(j-3)(j-2)(\phi^3/3!)u_j + \cdots \ , \tag{A7}$$

or, for $j \geq 5$,

$$A_j = \sum_{K=0}^{\infty} \frac{(j+K-5)!}{(j-5)!} u_{j+K-3} \frac{\phi^K}{K!} \ , \qquad A_{j+5} = \sum_{K=0}^{\infty} u_{j+2+K}\phi^K \ , \tag{A8}$$

and

$$B_j = \psi_{j-1} + (j-4)\phi\psi_j + (j-4)(j-3)(\phi^2/2!)\psi_{j+1} + (j-4)(j-3)(j-2)(\phi^3/3!)\psi_{j+2} + \cdots \ , \tag{A9}$$

or, for $j \geq 5$,

$$B_j = \sum_{K=0}^{\infty} \frac{(j+K-5)!}{(j-5)!} \psi_{j+K-1} \frac{\phi^K}{K!} \ , \qquad B_{j+5} = \sum_{K=0}^{\infty} \psi_{j+4+K}\phi^K \ . \tag{A10}$$

For the conserved densities A_j, we find that

$$A_0 = 12(\phi^4\phi_x)_x \ , \quad A_1 = -12(\phi^3\phi_x)_x \ , \quad A_2 = 12(\phi^2\phi_x)_x \ , \quad A_3 = -12(\phi\phi_x)_x \ , \quad A_4 = 12(\phi_x)_x \ . \tag{A11}$$

Thus, the densities $A_0 - A_4$ are trivial, i.e., gradients. Furthermore,

$$A_5 = \sum_{K=2}^{\infty} u_K\phi^{K-2} = \sum_{K=0}^{\infty} u_{K+2}\phi^K \tag{A12}$$

and

$$A_{5+j} = \frac{1}{j!}\frac{\delta^j}{\delta\phi^j} A_5 \ , \tag{A13}$$

where, for constant ϵ,

$$\frac{\delta}{\delta\phi} f(\phi) = \frac{d}{d\epsilon} f(\phi+\epsilon)|_{\epsilon=0} \ .$$

We note that

$$u = 12\frac{\partial^2}{\partial x^2}\ln\phi + A_5 \ . \tag{A14}$$

Appendix B: The Harry Dym Equation

From the KdV–MKdV equations we have obtained the equation for the singular surface:

$$\phi_t/\phi_x + \{\phi;x\} = \lambda \ , \tag{B1}$$

where

$$\{\phi;x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\frac{\phi_{xx}^2}{\phi_x^2}$$

is the Schwarzian derivative.

By the transform properties of the Schwarzian derivative,

$$\{\phi;x\} = -\phi_x^2\{x;\phi\} \tag{B2}$$

Now, under the change of variables;

$$x \to \phi \ , \qquad t \to t \ , \qquad \phi \to x \ , \tag{B3}$$

we note that

$$\phi_x = 1/x_\phi \ , \qquad x_t = \phi_t/\phi_x \ . \tag{B4}$$

Therefore, we find

$$x_\phi^2 x_t = \lambda x_\phi^2 + \{x;\phi\} \ , \tag{B5}$$

or, by expanding $\{x;\phi\}$,

$$x_t = \lambda - \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{x_\phi^2}\right) + \frac{3}{2}\left[\frac{\partial}{\partial\phi}\left(\frac{1}{x_\phi}\right)\right]^2 \ . \tag{B6}$$

Letting

$$v = 1/x_\phi \ , \tag{B7}$$

we find that

$$v_t = v^3 v_{\phi\phi\phi} \ . \tag{B8}$$

Finally, letting

$$v = cu^{-1/2} \ , \qquad -2c^3 = 1 \ , \tag{B9}$$

we find

$$u_t = \frac{\partial^3}{\partial\phi^3}u^{-1/2} \ , \tag{B10}$$

where

$$u = c^2/v^2 - c^2 x_\phi^2 \ . \tag{B11}$$

This equation is called the Harry Dym equation and is known to be integrable [14].

Equation (B8) raises some interesting questions concerning the nature of the Painlevé property. In general, about the singular manifold, $\psi = 0$,

$$V = \psi^{2/3} \sum_{j=0}^{\infty} V_j \psi^{j/3} \tag{B12}$$

with resonances at

$$j = -1, 2, 4 \ .$$

We note that $\ln\psi$ terms do not arise in the expansion for Eq. (B8).

(B12) demonstrates that Eq. (B8) does not posses the Painlevé property, as it has been defined herein, although the system (B8) is (presumably) integrable. The Painlevé property seems to be a sufficient, but not necessary condition, for integrability and would appear to require reformulation for systems with nontransformable branch point behavior.

Finally, we note that Eqs. (B5) and (B6) are invariant under the Moebius group acting on the independent variable ϕ. Hence, by restricting ϕ to a fundamental region in the complex ϕ plane it should be possible to define an $x = x(\phi)$ as an automorphic function [11]

$$x(\phi) = X\left(\frac{a\phi + b}{c\phi + d}\right) \tag{B13}$$

of the Moebius group.

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