APPLICATIONS OF THE SINGULAR MANIFOLD METHOD

John Weiss
Applied Mathematics Group,
49 Grand View Road, Arlington, MA 02476, USA
E-mail: johnweiss@rcn.com
Web: www.appmath.com

For systems without the Painlevé Property, it is possible to constrain the arbitrary functions in the expansion so as to restore the single-valued behavior. Depending on the number of constraints, the resulting expansion will represent a solution of reduced dimensions. The systems of constraints are expressed as a system of partial differential equations for the previously arbitrary functions in the expansion. We conjecture that the systems of constraint equations are completely integrable. Therefore, the analysis of differential equations by the Singular Manifold Method could lead to the discovery of new integrable systems. This is especially interesting for systems depending on many variables. We present some new results for the equations defined by the singular manifold analysis of the Sine-Gordon equation in several variables.

1 The Singular Manifold Method

For systems with the Painlevé Property, Bäcklund transformations appear as truncations of expansions of a solution about its singular manifold. With reference to the Lax pair for a system, these Bäcklund transformations are equivalent to transformations of linear systems developed by Laplace, Moutard and Darboux. The Painlevé analysis leads naturally to a reformulation of these systems in terms of the Schwarzian derivative. Using the conformal invariance of the Schwarzian derivative and certain discrete symmetries, a canonical form of Bäcklund transformations can be defined [1,2].

The Sine-Gordon equation

$$u_{xt} = \sin(u) \tag{1}$$

has the Painlevé property in the variable $V = e^{iu}$ where

$$VV_{xt} - V_x V_t = \frac{1}{2}(V^3 - V). \tag{2}$$

The expansion

$$V = \phi^{-2} \sum_{j=0}^{\infty} V_j \phi^j$$

with resonances $j = -1, 2$ and $V_0 = 4\phi_x\phi_t$ and $V_1 = \phi_{xt}$. The Painlevé transformation is

$$V = -4\frac{\partial^2}{\partial xt}\ln\phi + V_2 \tag{3}$$

where $V_2 = \phi_{xt}^2/(\phi_x\phi_t)$.

The Schwarzian modified equations are

$$\Omega_1 = \{\phi; t\} + 2Z_{tt}/Z = \alpha \tag{4}$$

$$\Omega_2 = \{\phi; x\} + 2W_{xx}/W = \beta \tag{5}$$

where $Z^2 = \phi_x/\phi_t$, $W^2 = \phi_t/\phi_x$, and $\alpha\beta = \frac{1}{4}$. The identity $\phi_x(\partial/\partial x)\Omega_1 + \phi_t(\partial/\partial t)\Omega_2 = 0$ demonstrates the equivalence of the above two equations. To find the *discrete symmetries,* let $\Theta = -\phi_{xt}/\phi_t$ and $\Phi = -\phi_{xt}/\phi_x$ and get the system

$$\Theta_t + \frac{1}{2}\Theta\Phi + \frac{\lambda}{2}\Theta/\Phi = 0 \tag{6}$$

$$\Phi_x + \frac{1}{2}\Theta\Phi + \frac{1}{2\lambda}\Phi/\Theta = 0. \tag{7}$$

The discrete symmetries of these equations imply the strong Bäcklund transformations for the Schwarzian equations.

$$\frac{\phi_{xt}}{\phi_t}\frac{\psi_{xt}}{\psi_t} = \frac{1}{\lambda}, \quad \phi_x\psi_x = 1 \tag{8}$$

$$\frac{\phi_{xt}}{\phi_x}\frac{\psi_{xt}}{\psi_x} = \lambda, \quad \phi_t\psi_t = 1 \tag{9}$$

$$\frac{\phi_{xt}}{\phi_t}\frac{\psi_{xt}}{\psi_t} = -\frac{1}{\lambda}, \quad \frac{\phi_{xt}}{\phi_x}\frac{\psi_{xt}}{\psi_x} = -\lambda \tag{10}$$

For instance, $\psi_t = -\phi_t^{-1}$ and $\psi_x = -(1/\lambda)\phi_{xt}^2/(\phi_t^2\phi_x)$ imply by the condition $\psi_{xt} = \psi_{tx}$ that ϕ satisfies the Schwarzian equation, (57).

The Moebius group is a point symmetry and composition with the BT above generates the solutions

$$\phi_0 = e^{\sigma t + x/\sigma}$$

$$\phi_1 = \tanh(\sigma t/2 + x/(2\sigma))$$

$$\phi_2 = \sinh(\sigma t + x/\sigma) + \sigma t - x/\sigma.$$

2 Conditional Integrability

For systems without single-valued expansions (Painlevé Property) , it is possible to constrain the *arbitrary* functions in the expansion so as to restore the single-valued behavior. Depending on the number of constraints the resulting expansion will represent a solution of reduced dimensions. The systems of constraints are expressed as a system of partial differential equations for the previously arbitrary functions (data) in the expansion. For this system of partial differential equations we make the following conjecture [3].

Conjecture: *The constraint equations are completely integrable.*

We consider the N dimensional elliptic Sine-Gordon equation [3]

$$-\triangle u = \sin u \tag{11}$$

where

$$\triangle = \sum \partial_{x_j}^2 = \nabla^t \nabla.$$

Using $V = e^{iu}$

$$-V \triangle V + \nabla V \cdot \nabla V = \frac{1}{2}(V^3 - V) \tag{12}$$

The Painlevé expansion

$$V = \phi^{-2} \sum_{j=0}^{\infty} V_j \phi^j \tag{13}$$

is valid with arbitrary V_2 iff the **Painlevé Condition**,

$$\Omega = \nabla\phi \cdot D \nabla\phi = 0, \tag{14}$$

where

$$D_{ii} = \frac{1}{2} \sum_{l=1, l\neq i}^{N} \sum_{m=1, m\neq i}^{N} (\phi_{lm}^2 - \phi_{ll}\phi_{mm}) \tag{15}$$

$$D_{ij} = \sum_{m=1}^{N} (\phi_{ij}\phi_{mm} - \phi_{im}\phi_{jm}). \tag{16}$$

The matrix D is symmetric and equation (86) is invariant under arbitrary scalings and translations in the independent variables, and orthogonal changes of independent variables

$$\nabla = B\nabla'$$

where $B^t = B^{-1}$. Using these properties it can be shown that the hypersurface M defined by the level sets

$$M = \{\hat{x}; \phi(\hat{x}) = \phi_0\}$$

has the property that principal curvatures of M as a manifold in R^N , $K_j; j = 1, \cdots, N-1$ verify the condition

$$K_1 K_2 + K_1 K_3 + \cdots + K_{N-2} K_{N-1} = 0. \tag{17}$$

This is the sum of the principal minors of order 2 of the second fundamental form of M. The condition of partial integrability requires the singular manifold to be an "Einstein space with null scalar curvature" [3].

When $N = 2$ the condition is trivial. When $N = 3$ the condition equation is

$$\begin{aligned} \phi_t^2(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + \phi_x^2(\phi_{tt}\phi_{yy} - \phi_{yt}^2) + \phi_y^2(\phi_{tt}\phi_{xx} - \phi_{xt}^2) \\ +2\phi_x\phi_t(\phi_{ty}\phi_{yx} - \phi_{xt}\phi_{yy}) \\ +2\phi_y\phi_t(\phi_{tx}\phi_{xy} - \phi_{yt}\phi_{xx}) \\ +2\phi_x\phi_y(\phi_{xt}\phi_{yt} - \phi_{xy}\phi_{tt}) = 0. \end{aligned} \tag{18}$$

Equation (18) may be integrated by a **Legendre transformation** [3],

$$\xi_1 = \phi_t,\ \xi_2 = \phi_x,\ \xi_3 = \phi_y,$$

$$t = W_{\xi_1},\ x = W_{\xi_2},\ y = W_{\xi_3},$$

$$\phi(t, x, y) + W(\xi_1, \xi_2, \xi_3) = t\xi_1 + x\xi_2 + y\xi_3. \tag{19}$$

The solution is

$$W = W_0 + W_1$$

where W_0 and W_1 are homogeneous of degree zero and one, respectively[3]. Again, the form of ϕ might be used to find integrable reductions.

When $N \geq 4$ it is not known if the condition is integrable. *Our conjecture states that it is integrable for all N.*

3 Some New Results for $N \geq 4$

The **Painlevé Condition** is invariant under arbitrary transformations $\phi = F(\theta)$, since,

$$\Omega = \sum_{i=1}^{N} \phi_{x_i}^4 \sum_{k>j} J_{x_j, x_k}\left(\frac{\phi_{x_k}}{\phi_{x_i}}, \frac{\phi_{x_j}}{\phi_{x_i}}\right) = 0 \tag{20}$$

where, $J_{x,y}(U, V) = U_x V_y - U_y V_x$.

There is a **Quadratic Solution** $\phi(\hat{x}) = \hat{x} \cdot A\hat{x}$ where

$$A = \begin{pmatrix} a_{11} & \mp 1 & \xi & \lambda \\ \mp 1 & a_{22} & \pm\lambda & \eta \\ \eta & \pm\lambda & -a_{11} & 1 \\ \lambda & \eta & 1 & -a_{22} \end{pmatrix} \tag{21}$$

$$a_{11} = \frac{1}{2}(\lambda - 1/\lambda)\xi \pm (\lambda + 1/\lambda)\eta,$$

$$a_{22} = \frac{1}{2}(\lambda + 1/\lambda)\xi \pm (\lambda - 1/\lambda)\eta,$$

$$\lambda^2 = -\frac{(\xi \mp \eta)^2}{4 + (\xi \pm \eta)^2}.$$

The **Painlevé Condition** is identical to

$$\Omega = \nabla\phi \cdot \left(- \left(\nabla\nabla^t\phi\right)^2 + \Delta\phi\nabla\nabla^t\phi + \Omega_2(\phi)I \right) \nabla\phi = 0 \tag{22}$$

where,

$$tr\left(\nabla\nabla^t\phi\right) = \Delta\phi = \lambda_1 + \cdots + \lambda_N,$$

$$\Omega_2(\phi) = \frac{1}{2}\left(tr\left(\left(\nabla\nabla^t\phi\right)^2 \right) - (\Delta\phi)^2 \right),$$

$$\Omega_2 = -\lambda_1\lambda_2 - \cdots - \lambda_{N-1}\lambda_N.$$

The **Characteristic Equation for** $\nabla\nabla^t\phi$ is:

$$\left(\nabla\nabla^t\phi\right)^N - \Omega_1\left(\nabla\nabla^t\phi\right)^{N-1} - \Omega_2\left(\nabla\nabla^t\phi\right)^{N-2} - \cdots\cdots - \Omega_N I = 0. \tag{23}$$

where

$$\Omega_j = (-1)^{j+1}(\lambda_1 \ldots \lambda_j + \cdots + and \lambda_{N-j+1} \ldots \lambda_N)$$

$$j\Omega_j = tr\left(\left(\nabla\nabla^t\phi\right)^j \right) - \sum_{k=1}^{j-1} \Omega_k tr\left(\left(\nabla\nabla^t\phi\right)^{j-k} \right) \tag{24}$$

The definition [4] of the **Legendre Transformation** implies:

$$\left(\nabla\nabla^t\phi\right)^2 - \Delta\phi\left(\nabla\nabla^t\phi\right) - \Omega_2 I = \Omega_N(\phi)\times$$

$$\left(\nabla\nabla^t w\right)^{N-2} - \Omega_1(w)\left(\nabla\nabla^t w\right)^{N-3} - \cdots - \Omega_{N-3}(w)\left(\nabla\nabla^t w\right) \tag{25}$$

since [4] $(\nabla\nabla^t\phi)(\nabla\nabla^t w) = I$ and

$$\Omega_j(w) = -\Omega_{N-j}(\phi)/\Omega_N(\phi).$$

Therefore,

$$\nabla\phi \cdot \left(\left(\nabla\nabla^t\phi\right)^2 - \Delta\phi\left(\nabla\nabla^t\phi\right) - \Omega_2 I \right) \nabla\phi = 0$$

implies the **Legendre Transform**

$$\hat{\xi} \cdot (\left(\nabla\nabla^t w\right)^{N-2} - \Delta w\left(\nabla\nabla^t w\right)^{N-3} - \cdots - \Omega_{N-3}\left(\nabla\nabla^t w\right))\hat{\xi} = 0 \qquad (26)$$

Let

$$\frac{d}{ds} = \hat{\xi} \cdot \nabla_{\hat{\xi}}.$$

For **N=3**, as found previously,

$$\hat{\xi}^t\left(\nabla\nabla^t w\right)\hat{\xi} = w_{ss} - w_s = 0$$

$$w = w_0 + w_1$$

$$\frac{d}{ds}w_0 = 0, \frac{d}{ds}w_1 = w_1$$

For **N=4**

$$\hat{\xi}^t \left(\left(\nabla\nabla^t w\right)^2 - \Delta w\left(\nabla\nabla^t w\right) \right) \hat{\xi} = \qquad (27)$$

$$\nabla(w_s - w) \cdot \nabla(w_s - w) - (w_{ss} - w_s)\Delta w = 0 \qquad (28)$$

Let w be homogeneous of degree m, i.e. $w_s = mw$. Then,

$$(m-1)^2 \, \nabla w \cdot \nabla w = m(m-1) \, w\Delta w$$

The substitution, with $m \neq 0, 1$,

$$w = \Theta^m,$$

represents the solution as a **Spherical Harmonics of degree one,**

$$\Delta\Theta = 0,$$

$$\Theta_s = \Theta.$$

From Hobson [5], the Spherical Harmonics of degree one can be found from the Spherical Harmonics of degree zero, $V = V(\theta, \phi, \beta)$. We define **Spherical Harmonics of degree zero** with $N = 4$ by

$$x = r\sin\theta\cos\phi \;\; y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta\cos\beta \;\; t = r\cos\theta\sin\beta,$$

$$\frac{\partial}{\partial \phi}\left(\frac{\cos\theta}{\sin\theta}V_\phi\right)+\frac{\partial}{\partial \beta}\left(\frac{\sin\theta}{\cos\theta}V_\beta\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta V_\theta\right)=0.$$

The change of variable,

$$\psi=\frac{1}{4}\log\tan\theta,$$

obtains

$$e^{-4\psi}\left(V_{\phi\phi}+V_{\psi\psi}\right)+e^{4\psi}\left(V_{\beta\beta}+V_{\psi\psi}\right)=0.$$

After a Fourier transform in (ϕ,β), find

$$V_{\psi\psi}=\left(\frac{t^2+s^2}{2}+\frac{t^2-s^2}{2}\tanh 4\psi\right)V.$$

4 Summary

A useful form is found for the Sine-Gordon, Painlevé Condition. Using a connection with a characteristic equation, the Legendre tranform for the Painlevé Condition is defined for an arbitrary dimension. For dimension four a representation of the solution is found in terms of the Legendre transform of spherical harmonics.

For arbitrary dimensions several new results have recently been developed and are being prepared for publication. In general, the Singular Manifold method and the Conditional Painlevé Property can define new integrable systems in several variables that may have application to the integration of analytic partial differential equations.

Acknowledgments

I thank Boris Konopelchenko, Marco Boiti, Flora Pempinelli, Giulio Solani, and Maria Concetta Gerardi, for the interesting and enjoyable Workshop. This work was supported in part by the Author Support Program at Waterloo Maple of Waterloo, Ontario. The Maple V program was used in this work.

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