Bäcklund transformations, Focal surfaces and the Two dimensional Toda lattice.

John Weiss
49 Grandview Road
Arlington, Mass 02476

Abstract

We find an infinite set of distinct Bäcklund transformations for the periodic two dimensional Toda lattice equations. These are directly related to the periodic fixed points of Bäcklund transformations for the KdV equation, Boussinesq equation, ect., and the *invariants* of the Laplace-Darboux transformations of *focal surfaces*.

The Two dimensional Toda lattice equations were first studied in the works of Laplace, Moutard and Darboux [1] in connection with their classification of surfaces and factorizations of linear differential operators. For instance, Moutard essentially solved the free end Toda lattice in the form of Wronskian determinants [1]. The two dimensional Toda lattice was derived by Laplace in the eighteenth century through results relating factorizations of linear differential operators with certain gauge invariants. This method was proposed by Darboux as a fundamental method for the classification of certain surfaces in space related as *focal*, or caustic surfaces.

In this letter we present some results for the two dimensional Toda lattice. These systems are known to be completely integrable systems [2,3]. A Bäcklund transformation for the two dimensional Toda lattice was found in reference [2]. However, for the periodic lattice, this Bäcklund transformation reduces the time-space dependence of the two dimensional Toda lattice to a traveling wave form. The Bäcklund transformation is then the factorization of the ordinary Toda syatem by the Kac-Van Moerbeke syatem. The direct search for a non-reductive Bäcklund transformation by the Painlevé method encounters formidable computational difficulties [4]. In reference [5] it is shown that the periodic Toda lattices are the *minus one* equations of certain hierarchies of integrable systems. The two component system (Sine-Gordon) is in the KdV sequence, the three component system is in the Boussinesq sequence, etc. Therefore, a general Bäcklund transformation for the periodic Toda lattice is of particular interest in the study of a wide class of integrable systems.

We will present an infinite set of distinct Bäcklund transformations for the two dimensional Toda latttice. Although simple to describe the resulting systems of integrable ordinary differential equations have a rich structure that depends strongly on the length and number theoretic properties of the period of the lattice. To our knowledge these are the first examples of non-reductive Bäcklund transformations for the two dimensional Toda lattice. We emphasize that the time-space dependence of the lattice is *factored* by commuting, finite dimensional hamiltonian flows. Surprisingly, these systems arise naturally from the periodic fixed points of the Bäcklund transformations for the Korteweg-de Vries equation.

By way of introduction we present the relevant results for the KdV system and the derivation of the Toda lattice from the *Laplace transformation* of focal surfaces. Then, we show how a simple generalization of the KdV result obtains the set of Bäcklund transformations for the Toda lattice.

The Korteweg-deVries equation [6] :

$$u_t + \frac{\partial}{\partial x}(u_{xx} + \frac{3}{2}u^2) = 0$$

has the Bäcklund transformation [6]

$$u = 4\frac{\partial^2}{\partial x^2}\ln\phi + u_2$$

where $u_2 = -(\phi_{xxx}/\phi_x) + \frac{1}{2}(\phi_{xx}/\phi_x)^2$ and

$$\frac{\phi_t}{\phi_x} + \{\phi; x\} = \lambda. \tag{1}$$

Equation (1) is somtimes known as the singular manifold equation for the KdV equation and itself has the two Bäcklund transformations [6]

$$\phi = \frac{a\psi + b}{c\psi + d} \tag{2}$$

$$ad - bc = 1$$

and

$$\phi_x = \psi_x^{-1} \tag{3}$$

where, for both transformations,

$$\frac{\psi_t}{\psi_x} + \{\psi; x\} = \lambda.$$

The expression

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2$$

is the Schwarzian derivative, which is invariant under the Moebius group (2) [7,8].

By itself, transformation (2) is a point symmetry that does not lead to new forms of solution, and transformation (3) by itself is in involution. The effective Bäcklund transformation (BT) for (1) is the composition of (2) and (3). We find that [9]:

$$\phi_{n+1,x} = \frac{\phi_n^2}{\phi_{n,x}} \tag{4}$$

$$\frac{\phi_{n+1,t}}{\phi_{n+1,x}} + \frac{\phi_{n,t}}{\phi_{n,x}} = \left(\frac{\phi_{n,xx}}{\phi_{n,x}}\right)^2 - 4\frac{\partial^2}{\partial x^2}\ln\phi_n + 2\lambda \tag{5}$$

is a BT for (1). The periodic fixed points of the BT are defined by equations (4) and (5) with:

$$n = 1, 2, 3, 4, \ldots \pmod{N}.$$

The periodic fixed points continue to define a strong BT for (1). That is, the integrability conditions

$$\phi_{n+1,xt} = \phi_{n+1,tx}$$

continue to inply that ϕ_n satisfy (1), and, by the periodicity $\pmod N$, the set $\{\phi_n, n = 1, 2, \ldots \pmod N\}$ are solutions of (1).

We have found [9] that if

$$\xi_j = \frac{\phi_{j,x}}{\phi_j}$$

then

$$\frac{\xi_{j+1,x}}{\xi_{j+1}} + \frac{\xi_{j,x}}{\xi_j} = \xi_j - \xi_{j+1}.$$

The KdV and Boussinesq systems are instances of the general system in component form [10]

$$\frac{\xi_{j,x}}{\xi_j} + \frac{\xi_{j+1,x}}{\xi_{j+1}} + \ldots + \frac{\xi_{j+p,x}}{\xi_{j+p}} = \xi_j - \xi_{j+p} \tag{6}$$

where $j = 1, 2, \ldots \pmod N$. The KdV systems correspond to $p = 1$ and the Boussinesq to $p = 2$. Let the circulant forward shift matrix [11] be

$$C = \mathrm{circ}[0, 1, 0, 0, \ldots, 0].$$

In the N-vector form equations (6) are

$$A \begin{pmatrix} \xi_{1,x}/\xi_1 \\ \vdots \\ \xi_{N,x}/\xi_N \end{pmatrix} = B\hat{\xi} \tag{7}$$

with

$$\begin{aligned} A &= I + C + \ldots + C^p \\ B &= I - C^p. \end{aligned}$$

The casimir integrals of (6) correspond to the null vectors of B. The null vectors of A produce the constraints.

Associated with the principal casimir, for any N

$$H_N = \prod_{j=1}^{N} \xi_j$$

we find the principal integrals of (6)

$$H_{N-pm-m} = L^m \circ H_N \tag{8}$$

,where $m = 0, 1, 2, \ldots$ and

$$L = \sum_{j=1}^{N} \partial_{\xi_j} \partial_{\xi_{j+1}} \ldots \partial_{\xi_{j+p}}.$$

The systems (7) have a Hamiltonian structure

$$A \begin{pmatrix} \xi_{1,x}/\xi_1 \\ \vdots \\ \xi_{N,x}/\xi_N \end{pmatrix} = B \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \nabla_{\hat{\xi}} H_1 \tag{9}$$

,where $H_1 = \sum_{j=1}^{N} \xi_j$.

The higher-order equations associated with the integrals (8) are

$$A \begin{pmatrix} \xi_{1,x}/\xi_1 \\ \vdots \\ \xi_{N,x}/\xi_N \end{pmatrix} = B \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \nabla_{\hat{\xi}} H_{N-pm-m}. \tag{10}$$

When A is invertible, then

$$\Omega = A^{-1} B$$

is an antisymmetric circulant matrix.

We have the systems

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_1 \tag{11}$$

and

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-pm-m} \tag{12}$$

where

$$M_{\hat{\xi}} = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \Omega \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix}$$

is the co-symplectic form.

Now, Darboux [1] has shown that the parameters (x, y) for surfaces in three dimensions can be defined so the coordinates (z_j) of the surface satisfy a partial differential equation of the form:

$$\partial^2 z/\partial x \partial y + a\partial z/\partial x + b\partial z/\partial y + cz = 0 \tag{13}$$

,where (a, b, c) are functionals of the first fundamental form in the (x, y) parameters.

Under the gauge transformation $z \to \lambda z$, the form of (13) is preserved and:

$$h = \partial a/\partial x + ab - c$$

$$k = \partial b/\partial y + ab - c$$

are invariant.

The *Laplace transformation* of a surface is a partial factorization of (13) in terms of the *invariants* [1].

$$z_1 = \partial z/\partial y + az$$

$$\partial z_1/\partial x + bz_1 = hz \tag{14}$$

Equations (14) imply that z satisfy (13) and z_1 satisfy the system

$$\partial^2 z_1/\partial x \partial y + a_1 \partial z_1/\partial x + b_1 \partial z_1/\partial y + c_1 z_1 = 0$$

where

$$a_1 = a - \partial \ln h/\partial y$$

$$b_1 = b$$

$$c_1 = c - \partial a/\partial x + \partial b/\partial y - b\partial \ln h/\partial y. \tag{15}$$

From (15) the Laplace transformation of the invariants is

$$h_1 = 2h - k - \partial^2 \ln h / \partial x \partial y$$

$$k_1 = h. \tag{16}$$

Darboux [1] studied the periodic fixed points of the Laplace transformation and found that these surfaces are related as a sequence of *focal surfaces*. From (16), the periodic fixed points are

$$\partial^2 \ln h_j / \partial x \partial y = -h_{j+1} + 2h_j - h_{j-1} \tag{17}$$

, where $j = 1, 2, 3, \cdots \pmod{n}$ and n is the order of the fixed point. The substitution

$$h_j = e^{\theta_{j+1} - \theta_j}$$

obtains the *two dimensional periodic Toda lattice*

$$\theta_{j,xy} = -e^{\theta_{j+1} - \theta_j} + e^{\theta_j - \theta_{j-1}} \tag{18}$$

We now find Bäcklund transformations for the *Darboux equations* (17) and the Toda lattice equations (18). With reference to systems (11) and (12), without loss of generality normalize the casimir, $H_N = 1$, and set

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \bigtriangledown_{\hat{\xi}} H_1 \tag{19}$$

$$\hat{\xi}_{,y} = M_{\hat{\xi}} \bigtriangledown_{\hat{\xi}} (H_{N-p-1} / H_N) \tag{20}$$

,where $H_{N-p-1} = L \circ H_N$. Then, let $\xi_j = e^{\psi_j - \psi_{j+1}}$ and find that (19), (20) imply

$$\psi_{j,xy} = e^{\psi_{j+p} - \psi_j} - e^{\psi_j - \psi_{j-p}} \tag{21}$$

,where $j = 1, 2, 3, \cdots \pmod{N}$.
To see this let $\xi_j = e^{\theta_j}$ and find

$$\hat{\theta}_{,x} = \Omega \bigtriangledown_\theta H_1$$

$$\hat{\theta}_{,y} = \Omega \bigtriangledown_\theta G$$

where

$$G = \sum e^{-\theta_j - \theta_{j+1} - \cdots - \theta_{j+p}}$$

$$= \sum 1/\xi_j \xi_{j+1} \cdots \xi_{j+p} = H_{N-p-1}/H_N.$$

It can be shown that

$$\hat{\theta}_{,xy} = C^{-p}(I - C^p)(I - C) \begin{pmatrix} e^{-\theta_1 - \theta_2 - \cdots - \theta_p} \\ \vdots \\ e^{-\theta_N - \theta_1 - \cdots - \theta_{p-1}} \end{pmatrix}.$$

Let $\hat{\theta} = (I - C)\hat{\psi}$ and find (21).

When $p = 1$ (21) are the Toda lattice of period N. If N and p are relatively prime (33) is again a Toda lattice of length N. If $N = mp$ (21) is p distinct lattices of length m. When N and p have common factors (21) there is one lattice for each distinct orbit of translation by $p \pmod N$. In all cases the set of fields ξ_j are directly related to the set of invariants h_j. When A is not invertible we find for equations (9) and (10) a similar connection with the Toda lattice. In this case one must take into account the *constraints* that apply to these systems to obtain a valid correspondence. See, for instance, reference [10].

Consideration of the form of (11), (12) and the possible relations between p and N determine that for a lattice of fixed length m there will exist an infinite sequence of distinct Bäcklund transformations. For instance, we have a Bäcklund transformation for a lattice of length m when $N = pm$ for $p = 1, 2, 3, \cdots$.

The Bäcklund transformation for the Toda lattice that was reported in ref. [2] corresponds in our formulation to the system (12) with $p = 1$

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-2}$$

$$\hat{\xi}_{,y} = -M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-2}.$$

Systems (19) and (20) have a rich structure. As a Bäcklund transformation a Toda lattice of fixed length these systems relate flows from different hierarchies of equations with the flows in the sequence through the Toda lattice. In other words, as described by (19) and (20) the hierarchies of flows through KdV, Boussinesq, etc. are interelated.

References

[1] G. Darboux, *Théorie Générale des Surfaces, II* Chelsea Publishing Company, New York (1972).

[2] A.P. Fordy and J. Gibbons, *Comnmun. Math. Phys.* **77** (1980), p. 2.

[3] A.V. Mikhailov, *Physica* **8D** (1981) p. 73.

[4] J. Weiss, *J. Math. Phys.* **27**(1986) p. 1293.

[5] B.A. Kuperschmidt and George Wilson, *Commun. Math. Phys.* **81** (1981) p. 189.

[6] J.Weiss, *J.Math.Phys.* **25** (1984), p. 13.

[7] E.Hille, *Ordinary Differential Equations in the Complex Plane*, Wiley, New York (1976).

[8] J.Weiss, *J.Math.Phys.* **24**, (1983) p. 1405.

[9] J.Weiss, *J.Math.Phys.* **27** (1986) p. 2647.

[10] J.Weiss, *J.Math.Phys.* **28** (1987) p. 2025.

[11] P.Davis, *Circulant Matrices*, Wiley, New York (1983).